Partial Differential Equations for Pricing

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Partial differential equations (PDEs) play a crucial role in the field of financial mathematics, particularly in the pricing of options. The Black-Scholes equation, a well-known PDE, forms the foundation for modern option pricing theory. It describes the evolution of the option's price as a function of the underlying asset price, time, and other relevant parameters. By solving the Black-Scholes PDE, one can derive analytical solutions for the prices of European call and put options, providing insights into their fair values. Additionally, PDEs are used to model various boundary conditions and to address more complex financial instruments, such as American options and exotic derivatives. Numerical methods, such as finite difference methods, are often employed to solve these PDEs, enabling practitioners to handle real-world complexities and market conditions. This application of PDEs in option pricing underscores their significance in ensuring accurate and robust financial models.

Contents

0 Introduction

Risk-neutral pricing and partial differential equations (PDEs) are connected through the Feynman-Kac theorem, which provides a bridge between stochastic processes and PDEs. Here's how they are connected:

- **Risk-Neutral Measure** In risk-neutral pricing, we assume that the expected return of the underlying asset is the risk-free rate. This allows us to price derivatives by discounting the expected payoff under the risk-neutral measure.
- **Stochastic Differential Equations (SDEs)**: The dynamics of the underlying asset prices are modeled using SDEs which are as follows:

$$\mathrm{d}X(u) = \underbrace{\beta(u, X(u))}_{\text{drift}}\,\mathrm{d}u + \underbrace{\gamma(u, X(u))}_{\text{diffusion}}\,\mathrm{d}W(u) \text{ with } \underbrace{X(t) = x}_{\text{initial condition}} \tag{SDE}$$

  It is emphasized that the Markov property is a fundamental characteristic of solutions to SDEs where the only randomness we permit on the right-hand side of (SDE) is the randomness inherent in the solution $X(u)$. Markov property states that the future behavior of a stochastic process depends only on its current state, independent of its past history. In the context of SDEs, this property implies that given the current value of the process at a certain time, its future evolution is determined solely by its current state and is unaffected by how it reached that state.

- **Feynman-Kac Theorem** This theorem states that the price of a derivative can be represented as the solution to a certain PDE. Specifically, if $f(t, S)$ is the price of the derivative at time t when the underlying asset price is S which follows $\mathrm{d}S(t) = rS(t)\mathrm{d}t + \sigma S(t)\mathrm{d}\tilde{W}(t)$, then $f(t, S)$ solves the following PDE:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

  with the terminal condition $f(T, S) = h(S)$, where $h(S)$ is the payoff at maturity T.

- **Boundary Conditions** The terminal and boundary conditions of the PDE are determined by the specific payoff structure of the derivative. For example, for a European call option with strike price K, the terminal condition is $f(T, S) = \max(S - K, 0)$.

By solving the PDE with appropriate initial and boundary conditions, we can determine the price of the derivative at any time before maturity. Thus, risk-neutral pricing, which relies on expected discounted payoffs, can be translated into solving a PDE, providing a powerful tool for pricing a wide range of derivative securities.

1 Example of SDEs

1.1 One-dimensional linear

Consider the following SDE

$$\mathrm{d}X(u) = (a(u) + b(u)X(u))\,\mathrm{d}u + (\gamma(u) + \sigma(u)X(u))\,\mathrm{d}W(u) \tag{One-dimensional linear}$$

Define

$$Z(u) = \exp\left(\int_t^u \sigma(v)\mathrm{d}W(v) + \int_t^u \left(b(v) - \tfrac{1}{2}\sigma^2(v)\right)\mathrm{d}v\right)$$

$$Y(u) = X(t) + \int_t^u \frac{a(v) - \sigma(v)\gamma(v)}{Z(v)} \mathrm{d}v + \int_t^u \frac{\gamma(v)}{Z(v)} \mathrm{d}W(v)$$

Then $X(u) = Y(u)Z(u)$ solves (One-dimensional linear). To ensure the Markov property, $a(u), b(u), \gamma(u)$, and $\sigma(u)$ are non-random.

1.2 Geometric Brownian Motion

Geometric Brownian Motion (GBM) are versatile in modeling the dynamics of financial asset prices.

$$\mathrm{d}S(u) = \alpha S(u)\mathrm{d}u + \sigma S(u)\mathrm{d}W(u) \tag{GBM}$$

A closed-form solution to (GBM) is as follows

$$S(T) = S(t) \cdot e^{\sigma(W(T)-W(t))+\left(\alpha-\frac{1}{2}\sigma^2\right)(T-t)}.$$

1.3 Hull-White Model

The Hull-White model is a single-factor interest rate model that describes the evolution of short-term interest rates. It is an extension of the Vasicek model and is used to model the future movements of interest rates with greater flexibility. The model is expressed as the following SDE:

$$\mathrm{d}R(u) = (a(u) - b(u)R(u))\mathrm{d}u + \sigma(u)\mathrm{d}\tilde{W}(u) \tag{Hull-White}$$

This SDE has a closed-form solution as follows:

$$R(T) = re^{-\int_t^T b(v)\mathrm{d}v} + \int_t^T e^{-\int_t^T b(v)\mathrm{d}v}\alpha(u)\mathrm{d}u + \int_t^T e^{-\int_t^T b(v)\mathrm{d}v}\sigma(u)\mathrm{d}\tilde{W}(u)$$

Since Itô integrals with deterministic functions are normally distributed, $R(T)$ is normally distributed and thus it takes negative values with positive probability.

1.4 Cox-Ingersoll-Ross

The Cox-Ingersoll-Ross (CIR) model is another mathematical model used to describe the evolution of interest rates over time. It is represented by the following SDE

$$\mathrm{d}R(u) = (a - bR(u))\mathrm{d}u + \sigma\sqrt{R(u)}\mathrm{d}\tilde{W}(u) \tag{CIR}$$

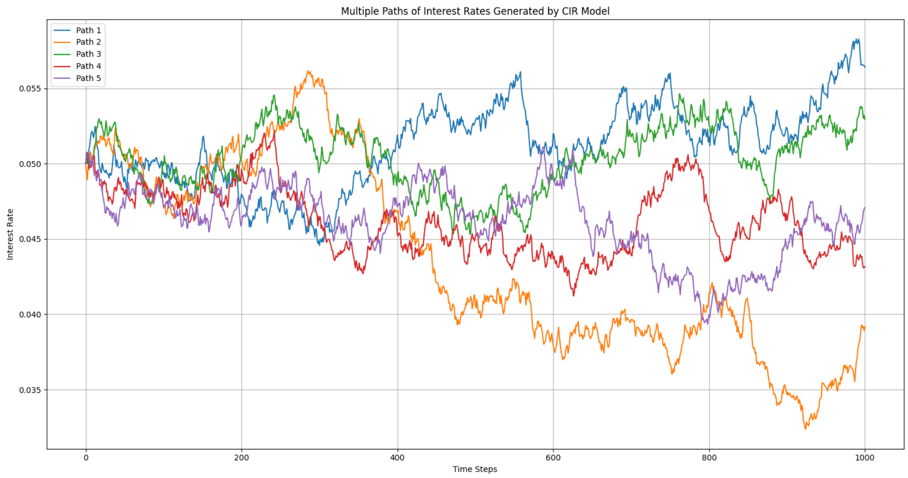
The CIR model ensures non-negative interest rates and is widely used in the pricing of fixed-income securities and interest rate derivatives. Notably, moment generating function (MGF) of $R(u)$ is obtained using Ornstein-Uhlenbeck SDE:

$$\mathrm{d}X_j(t) = -\frac{b}{2}X_j(t)\mathrm{d}t + \frac{1}{2}\sigma\mathrm{d}W_j(t) \tag{Ornstein-Uhlenbeck}$$

Let $R(t) = \sum_{j=1}^d X_j^2(t)$. Then $B(t) = \sum_{j=1}^d \int_0^t \frac{X_j(s)}{\sqrt{R(s)}}\mathrm{d}W_j(s)$ is a Brownian motion and

$$\mathrm{d}R(u) = (a - bR(u))\,\mathrm{d}u + \sigma\sqrt{R(u)}\mathrm{d}B(u)$$

Figure 1.4 illustrates possible interest rate path generate using CIR model.



2 Feynman-Kac Theorem

Using the Feynman-Kac theorem to price options involves deriving a PDE that models the option's value based on the stochastic behavior of the underlying asset's price. This PDE expresses the option price as the expected discounted value of its future payoff, accounting for the asset's price dynamics and volatility. This method provides a robust framework for accurately pricing options in various market conditions.

2.1 Black-Scholes-Merton Model

Suppose the asset price $S(t)$ satisfies

$$\mathrm{d}S(t) = rS(t)\mathrm{d}t + \sigma S(t)\mathrm{d}\tilde{W}(t)$$

Consider an option on $S(t)$ with payoff $h(S(T))$ at maturity T *e.g.,* $h(x) = (x - K)^+$ and $h(x) = (K - x)^+$ for call and put options respectively. Feynman-Kac Theorem then produces BSM model.

2.2 Zero-Coupon Bonds

Suppose that the interest rate $R(t)$ satisfies

$$\mathrm{d}R(u) = \beta(u, R(u))\mathrm{d}u + \gamma(u, X(u))\mathrm{d}W(u)$$

Moreover, recall that the discount process is defined as below

$$D(t) = e^{-\int_0^t R(s)\mathrm{d}s}$$

The following then is true:

$$D(t)B(t,T) = \tilde{\mathbb{E}}\left[D(T)|\mathcal{F}(t)\right]$$

Define $f(t, R(t)) := B(t, T)$. Discounted Feynman-Kac gives

$$f_t(t,r) + \beta(t,r) f_r(t,r) + \frac{1}{2}\gamma^2(t,r) f_{rr}(t,r) = r f(t,r) \text{ where } f(T,r) = 1, \forall r.$$

Similarly, we can obtain a PDE for pricing options on bonds. Fix $0 \le t \le T_1 \le T_2$. The price of a call option with expiry T_1 to buy a bond with expiry at T_2 satisfies the following

$$c(t, R(t)) = \tilde{\mathbb{E}}\left[e^{-\int_t^T R(s) \mathrm{d}s} \cdot (f(T_1, R(T_1)) - k)^+ \right]$$

Feynman-Kac Theorem provides

$$c_t(t,r) + \beta(t,r) c_r(t,r) + \frac{1}{2}\gamma^2(t,r) c_{rr}(t,r) = r c(t,r) \text{ where } c(T_1, r) = (f(T_1, r) - K)^+, \forall r.$$

This is the same PDE as for bond prices only with different terminal conditions.

3 Python Codes

3.1 CIR Model Simulator

```
import numpy as np
import matplotlib.pyplot as plt

def generate_cir_rates(r0, a, b, sigma, T, num_steps, num_paths):
    dt = T / num_steps
    rates = np.zeros((num_steps + 1, num_paths))
    rates[0, :] = r0

    for i in range(num_steps):
        dW = np.sqrt(dt) * np.random.normal(0, 1, num_paths)
        rates[i+1, :] = rates[i, :] + a * (b - rates[i, :]) * dt + sigma * np.sqrt
                                                    (rates[i, :]) * dW
        # note the formula looks a bit different from what we have in the text.
    return rates
```