Probability Theory

$\underbrace{\mu_{X,Y}(C)}_{\text{joint dist. measure}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_C(x,y)\underbrace{f_{X,Y}(x,y)}_{\text{Joint dist.}}\mathrm{d}x\mathrm{d}y$ Ex:
$X=Y\sim Unif[0,1] \rightsquigarrow (X,Y)$ has no joint den. func.
Marginal dist. measures $\mu_X(A)=\mu_{X\times Y}(A\times\mathbb{R})$
Indp. $\iff$ j dist. mea. f $\iff$ j cum. dist fncs f $\iff$ mgf functions f $\iff$ if j density exists $\rightsquigarrow$ j density fncs f $\Rightarrow$ exp. f
Ex: $X\sim N(0,1)$ & $Y=XZ$ & Z R. X,Y uncrr& not indp.
$X(t)$ is GP if $X(t_1),\cdots,X(t_n)$ are j normally dis.

$$m(t)=\mathbb{E}X(t), c(s,t)=\mathrm{Cov}(X(s),X(t))$$

Ex: $I(t)=\int_0^t\Delta(u)\mathrm{d}W(u)$ is GP if $\Delta(u)$ is deter.
Reflection principle

$$m>0, w\le m, \mathbb{P}(\tau_m\le t, W(t)\le w)=\mathbb{P}(W(t)\ge 2m-w)$$

$\rightsquigarrow$ joint density $(M(t),W(t))$ with $M(t)=\max W(s)$

Stochastic Calculus

$\Delta(t)$ stoch. pro. ad. to fil. gen. by $W(t)$

Itô Int. $I(t)=\int_0^t\Delta(u)\mathrm{d}W(u)\rightsquigarrow$ mart., $\mathbb{E}I^2(t)=\mathbb{E}\int_0^t\Delta^2(u)\mathrm{d}u$

$$QV:[I,I](t):=\int_0^t\Delta^2(u)\mathrm{d}u,\ \mathrm{d}I(t)=\Delta(t)\mathrm{d}W(t)$$

$$\text{Itô process: } X(t)=X(0)+\int_0^t\Delta(u)\mathrm{d}W(u)+\int_0^t\Theta(u)\mathrm{d}u$$

$$\mathrm{d}f(t,X(t))=f_t\mathrm{d}t+f_x\mathrm{d}X+\tfrac{1}{2}f_{xx}\mathrm{d}X\mathrm{d}X$$

Levy M mart., cont. path, $M(0)=0, \mathrm{d}M\mathrm{d}M=\mathrm{d}t.M\rightsquigarrow BM$.

$\mathrm{d}S(t)=\alpha S(t)\mathrm{d}t+\sigma S(t)\mathrm{d}W(t), S(t)=S(0)e^{\sigma W(t)+(\alpha-\frac{\sigma^2}{2})t}$
$\mathrm{d}X(t)=\Delta(t)\mathrm{d}S(t)+r(X(t)-\Delta(t)S(t))\mathrm{d}t$
Agent pays $C(t)\rightsquigarrow \mathrm{d}X=\Delta\mathrm{d}S+R\left(X-\Delta S\right)\mathrm{d}t-C\mathrm{d}t$

$$\mathrm{d}X=\underbrace{rX\mathrm{d}t}_{\text{average rate of return}}+\underbrace{\Delta(\alpha-r)S\mathrm{d}t}_{\text{risk premium}}+\underbrace{\Delta\sigma S\mathrm{d}W}_{\text{volatility term}}.$$

Dividend $\mathrm{d}S(t)=\alpha(t)S(t)\mathrm{d}t+\sigma(t)S(t)\mathrm{d}W(t)-A(t)S(t)\mathrm{d}t$
Multidim market model $\mathrm{d}\left(DX\right)=\sum_{i=1}^m\frac{\Delta_i}{D}\mathrm{d}\left(DS_i\right)$.
$\mathrm{d}S_i(t)=\alpha_i(t)S_i(t)+S_i(t)\sum_{j=1}^d\sigma_{ij}(t)\mathrm{d}W_j(t),\quad\forall i\in[1,m]$
Cost of carry $\mathrm{d}X=\Delta\mathrm{d}S-a\Delta\mathrm{d}t+r(X-\Delta S)\mathrm{d}t$
Here $\mathrm{d}S(t)=rS(t)\mathrm{d}t+\sigma S(t)\mathrm{d}\tilde{W}(t)+a\mathrm{d}t$
Cash flow valuation $\mathrm{d}X(u)=\mathrm{d}C(u)+r(u)X(u)\mathrm{d}u$
Risk-neutral pricing $e^{-rt}V(t)=\tilde{\mathbb{E}}[e^{-rT}V(T)|\mathcal{F}(t)]$
Forward & future With constant interest rate r

$$\tilde{\mathbb{E}}\left[e^{-r(T-t)}\left(S(T)-K\right)|\mathcal{F}(t)\right]=0\iff K=\tilde{\mathbb{E}}\left[S(T)|\mathcal{F}(t)\right]$$

Black Scholes Model

$$\mathrm{d}e^{-rt}X(t)=\mathrm{d}e^{-rt}c(t,S(t))\Rightarrow\Delta(t)=c_x(t,S(t))$$

$$c_t+rxc_{xx}+\tfrac{1}{2}\sigma^2x^2c_{xx}=rc$$

$$c(T,x)=(x-K)^+, c(t,0)=0, 0=\lim_{x\to+\infty}c(t,x)-(x-e^{-r(T-t)}K)$$

$$c(t,x)=xN(d_+)-Ke^{-r(T-t)}N(d_-), d_\pm=\tfrac{1}{\sigma\sqrt{\tau}}[\log\tfrac{x}{K}+(r\pm\tfrac{\sigma^2}{2})\tau]$$

$$P\&L=\mathrm{d}(e^{-rT}P)-\frac{\partial P}{\partial S}\mathrm{d}(e^{-rT}S)=e^{-rT}[\tfrac{1}{2}\frac{\partial^2P}{\partial S^2}(\mathrm{d}<S>-\sigma^2S^2\mathrm{d}t)]$$

$$\mathrm{d}P=\underbrace{\frac{\mathrm{d}P}{\mathrm{d}t}}_{:=\Theta}*+\underbrace{\frac{\mathrm{d}P}{\mathrm{d}S}}_{:=\Delta}*+\underbrace{\frac{\mathrm{d}P}{\mathrm{d}\sigma}}_{:=v}*+\underbrace{\frac{\mathrm{d}P}{\mathrm{d}r}}_{:=\rho}*+\tfrac{1}{2}\underbrace{\frac{\mathrm{d}^2P}{\mathrm{d}S^2}}_{:=\Gamma}*$$

$$+\tfrac{1}{2}\underbrace{\frac{\mathrm{d}^2P}{\mathrm{d}S\mathrm{d}\sigma}}_{:=\text{Vanna}}*+\tfrac{1}{2}\underbrace{\frac{\mathrm{d}^2P}{\mathrm{d}\sigma^2}}_{:=\text{Volga}}*+\tfrac{1}{2}\underbrace{\frac{\mathrm{d}^2P}{\mathrm{d}S\mathrm{d}t}}_{:=\text{Charm}}*+\text{re}$$

PDE

$$\mathrm{d}X(u)=\beta(u,X(u))\mathrm{d}u+\gamma(u,X(u))\mathrm{d}W(u), X(t)=x.$$

$$\mathrm{d}R(u)=(a(u)-b(u)R(u))\mathrm{d}u+\sigma\mathrm{d}\tilde{W}(u),\ \text{Vasicek, HW}$$

$$\mathrm{d}R(u)=(a(u)-b(u)R(u))\mathrm{d}u+\sigma\sqrt{R(u)}\mathrm{d}\tilde{W}(u),\ \text{CIR}$$

$$\mathrm{d}X_j=-\tfrac{b}{2}X_j(t)\mathrm{d}t+\tfrac{\sigma}{2}\mathrm{d}W_j(t), R(t)=\sum_{j=1}^dX_j^2(t)\rightsquigarrow CIR$$

HJM

HJM has zero-coupon bond with maturity $T,\forall T\in[0,\bar{T}]$.

$$\mathrm{d}f(t,T)=\alpha(t,T)\mathrm{d}t+\sigma(t,T)\mathrm{d}W(t), 0\le t\le T$$

Term-structure model satis. HJM no-arbitrage if forward rates

$$\mathrm{d}f(t,T)=\sigma(t,T)\sigma^*(t,T)\mathrm{d}t+\sigma(t,T)\mathrm{d}\tilde{W}(t)$$

$$\mathrm{d}D(t)B(t,T)=-\sigma^*(t,T)D(t)B(t,T)\mathrm{d}\tilde{W}(t),$$

$$\sigma^*(t,T)=\int_t^T\sigma(t,v)\mathrm{d}v$$

Every term-structure model driven by BM is HJM.

$$B(t,T)=e^{-\int_t^Tf(t,v)\mathrm{d}v}, f(t,T)=-\frac{\partial}{\partial T}\log B(t,T)$$

Two Factor Models

$$\mathrm{d}X_1(t)=(a_1-b_{11}X_1(t)-b_{12}X_2(t))\mathrm{d}t+\sigma\mathrm{d}\tilde{B}_1(t)$$

$$\mathrm{d}X_2(t)=(a_2-b_{21}X_1(t)-b_{22}X_2(t))\mathrm{d}t+\sigma\mathrm{d}\tilde{B}_2(t)$$

$R(t)=\epsilon_0+\epsilon_1X_1(t)+\epsilon_2X_2(t)$ **Vasicek**

$$\mathrm{d}Y_1(t)=(\mu_1-\lambda_{11}Y_1(t)-\lambda_{12}Y_2(t))\mathrm{d}t+\sqrt{Y_1(t)}\mathrm{d}\tilde{W}_1(t)$$

$$\mathrm{d}Y_2(t)=(\mu_2-\lambda_{21}Y_1(t)-\lambda_{22}Y_2(t))\mathrm{d}t+\sqrt{Y_2(t)}\mathrm{d}\tilde{W}_2(t)$$

$R(t)=\delta_0+\delta_1Y_1(t)+\delta_2Y_2(t)$ **CIR**

$f(t,Y_1(t),Y_2(t))=B(t,T)\rightsquigarrow$ set $\mathrm{d}t$-term in $\mathrm{d}D(t)B(t,T)=0$.
Solve PDE4 $f(t,y_1,y_2)=e^{-y_1C_1(T-t)-y_2C_2(T-t)-A(T-t)}$.

Forward LIBOR $L(t,T)$

$1+\delta L(t,T)=\frac{B(t,T)}{B(t,T+\delta)}$. Price $L(T,T)$ at $t:B(t,T+\delta)L(t,T)$
$B(t,T)=\tilde{\mathbb{E}}\left[e^{-\int_t^TR(s)\mathrm{d}s}|\mathcal{F}(t)\right]$
Backs. L. $L(t,T)B(t,T+\delta)$ P. at $t\le T$, Pay $L(T,T)$ at $T+\delta$
Black-C. $T+\delta\rightsquigarrow(L(T,T)-K)^+.\frac{\mathrm{d}L(t,T)}{L(t,T)}=\gamma(t,T)\mathrm{d}\tilde{W}^{T+\delta}(t)$

$$B(0,T+\delta)[L(0,T)N(d_+)-KN(d_-)], d_\pm=\frac{1}{\sqrt{\Gamma}}[\log\frac{L(0,T)}{K}+\Gamma]$$

$\Gamma=\int_0^T\gamma^2(t,T)\mathrm{d}t,\ \gamma(t,T)=\frac{1+\delta L(t,T)}{\delta L(t,T)}[\sigma^*(t,T+\delta)-\sigma^*(t,T)]$
Forward LIBOR, $T+\delta$ & T-maturity zero-coupon bonds vols.

Numeraire

Asset representation N primary or derivative, no dividend
$\mathrm{d}N=rN\mathrm{d}t+N\boldsymbol{\nu}\cdot\mathrm{d}\tilde{\boldsymbol{W}}\rightsquigarrow N=N(0)e^{\int_0^t\boldsymbol{\nu}\cdot\mathrm{d}\tilde{\boldsymbol{W}}+\int_0^t(R-\frac{1}{2}||\boldsymbol{\nu}||^2)\mathrm{d}u}$
$\tilde{\boldsymbol{W}}_j^{(N)}=-\int_0^t\boldsymbol{\nu}_j\mathrm{d}u+\tilde{\boldsymbol{W}}_j,\tilde{\mathbb{P}}^{(N)}(A)=\frac{1}{N(0)}\int_AD(T)N(T)\mathrm{d}\tilde{\mathbb{P}}$
$\mathrm{d}DS=DS\boldsymbol{\sigma}\cdot\mathrm{d}\tilde{\boldsymbol{W}},\mathrm{d}DN=DN\boldsymbol{\nu}\cdot\mathrm{d}\tilde{\boldsymbol{W}},\frac{\mathrm{d}S^N}{S^N}=[\boldsymbol{\sigma}-\boldsymbol{\nu}]\cdot\mathrm{d}\tilde{\boldsymbol{W}}^N$
Ex: $\frac{\mathrm{d}S}{S}=r\mathrm{d}t+\sigma\mathrm{d}\tilde{W}_1,\frac{\mathrm{d}N}{N}=r\mathrm{d}t+\nu\mathrm{d}\tilde{W}_3,V^2\frac{S}{N}=\sigma^2-2\rho\sigma\nu+\nu^2$
T-for. $\frac{V(t)}{B(t,T)}=\tilde{\mathbb{E}}^T[V(T)|\mathcal{F}(t)]$, $\mathrm{d}F_S(t,T)=\sigma F_S(t,T)\mathrm{d}\tilde{W}^T$
Rnd int rate $V(t)=S(t)N(d_+(t))-KB(t,T)N(d_-(t))$

Exotic Options

Perp. Am. put $\mathrm{d}e^{-rt}v_{L_*}(S(t))$ is supermart.
$v(t,x)=\max_{\tau\in\mathcal{T}_{t,T}}\tilde{\mathbb{E}}\left[e^{-r(\tau-t)}\left(K-S(\tau)\right)|S(t)=x\right]$
Am. call $h\ge0$ cvx. Discnt. intr. val $e^{-rT}h(S(t))$ is submart.
Div paying Am. call Opt. ex.: right before div payment

SVs

SABR $\mathrm{d}F_t=\alpha_tF_t^\beta\mathrm{d}W_t^1,\mathrm{d}\alpha_t=\nu\alpha_t\mathrm{d}W_t^2,\mathrm{d}W_t^1\mathrm{d}W_t^2=\rho\mathrm{d}t$
Heston $\frac{\mathrm{d}S_t}{S_t}=\mu\mathrm{d}t+\sqrt{\nu_t}\mathrm{d}W_t^{1,P},\mathrm{d}\nu_t=$
$\kappa(\theta-\nu_t)+\xi\sqrt{\nu_t}\mathrm{d}W_2^{2,P},\mathrm{d}W_t^{1,P}\mathrm{d}W_2^{2,P}=\rho\mathrm{d}t$

Jump Process

$f_\tau(t)=\lambda e^{-\lambda t},\mathbb{E}\tau=\frac{1}{\lambda},F=1-e^{-\lambda t},\mathbb{P}(\tau>t+s|\tau>s)=e^{-\lambda t}$
n^{th} jump: $\tau_1+\cdots+\tau_n$. $\mathbb{E}\tau_i=\frac{1}{\lambda}$. Arr. times: $S_n=\sum_{k=1}^n\tau_k$
Poisson process $N(t)=\#$ jumps before t. Intensity $=\lambda$.
Density $=\frac{(\lambda t)^k}{k!}e^{-\lambda t},N(t+s)-N(t)\sim N(s),N(t)-\lambda t$ mart.
$\mathbb{E}N(t)-N(s)=\lambda(t-s),\mathrm{Var}\,N(t)-N(s)=\lambda(t-s)$
Compound Pois. proc. $Q(t)=\sum_{k=1}^{N(t)}Y_i,Y_k$ iid, $\mathbb{E}Y_i=\beta$.
$\mathbb{E}Q(t)-Q(s)=\lambda\beta(t-s),\mathrm{Var}\,Q(t)-Q(s)=\lambda\beta(t-s),Q(t)-\lambda\beta t$ mart. $\varphi_{Q(t)}(u)=\exp(\lambda t(\varphi_Y(u)-1))$
Decom. $\mathbb{P}(Y_k=y_j)=p_j,\forall j\in[1,M]$. $N_1,\cdots,N_M$ indep.
Poi. pr. $\mathbb{E}N_k=\frac{1}{\lambda p_k}$. $Q(t)=\sum_{m=1}^My_mN_m(t)$

$$\underbrace{X(t)}_{\text{jump process}}=\underbrace{X(0)}_{\text{nonrandom}}+\underbrace{I(t)}_{=\int_0^t\Gamma(s)\mathrm{d}W(s)}+\underbrace{R(t)}_{=\int_0^t\Theta(s)\mathrm{d}s}+J(t)$$

$J(t)$val. imm. a j. $J(t-)$val. imm. b j. $\Delta J(t)=J(t)-J(t-)$

$$\int_0^t\Phi(s)\mathrm{d}X(s)=\int_0^t\Phi(s)\Gamma(s)\mathrm{d}W(s)+\int_0^t\Phi(s)\Theta(s)\mathrm{d}s+\sum_{0<s\le t}\Phi(s)\Delta J(s)$$

$\Phi(s)$ left-cont.(predictable) $\rightsquigarrow$ mart.
QV $[X_i,X_j](T)=\int_0^T\Gamma_i(s)\Gamma_j(s)\mathrm{d}s+\sum_{0<s\le T}\Delta J_i(s)\Delta J_j(s)$
Itô $f(X(t))=f(X(0))+\int_0^tf'(X(s))\mathrm{d}X^c(s)+$
$\frac{1}{2}f''(X(s))\mathrm{d}X^c(s)\mathrm{d}X^c(s)+\sum_{0<s\le t}[f(X(s))-f(X(s-))]$
W, N indep. $W(t)$ & $N(t)$ indep. defined on same prob. sp.
Doleans-Dade $Z^X=\exp^{X^c-\frac{1}{2}[X^c,X^c]}\prod_{0<s\le t}(1+\Delta(s))$