Dupire's Formula

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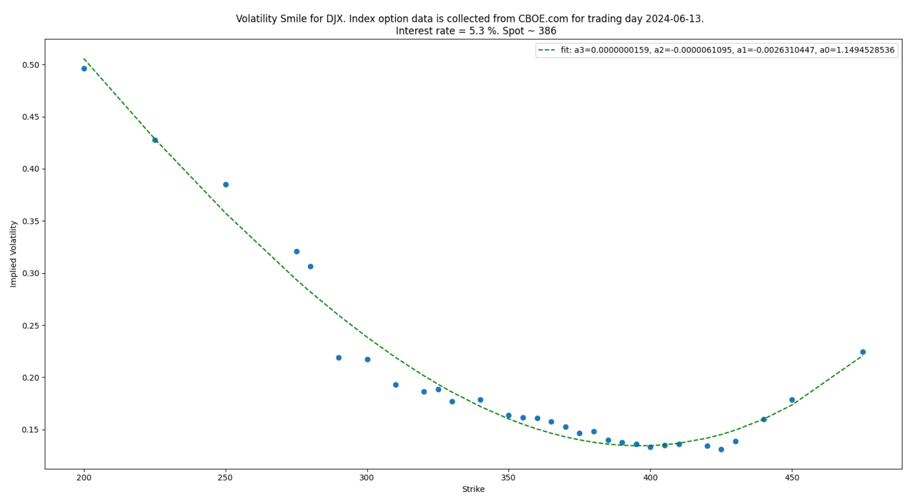
1 Background

In BSM, assets' prices are modeled as log-normal random variables and volality is assumed constant:

$$\sigma = \sigma_{\text{implied}}$$

Implied volatilities are computed using market quotes of options in BSM pricing model and they vary across different option contracts. In particular, volatility is not constant across strikes.



To capture volatility skew, we may assume that volatility itself is a random variable. Local volatility of the underlying asset is a deterministic function of asset price and time t *i.e.,*

$$\sigma_t = \sigma(t, S(t))$$

Dupire's formula provides a closed-form expression for local volatility in the event of calibration using market prices of European call options. In detail, suppose that a stock is governed by the following SDE

$$\mathrm{d}S(u) = rS(u)\mathrm{d}t + \sigma(u, S(u))S(u)\mathrm{d}\tilde{W}(u)$$

Denote by $\tilde{p}(t, T, x, y)$ the transition density. In particular, time-zero price of a call expiring at time T when $S(0) = x$ is equal to

$$c(0, T, x, K) = e^{-rT} \int_K^{+\infty} (y - K)\tilde{p}(0, T, x, y)\mathrm{d}y.$$

Dupire's formula then asserts

$$\boxed{c_T(0, T, x, K) = -rKc_K(0, T, x, K) + \frac{1}{2}\sigma^2(T, K)K^2 c_{KK}(0, T, x, K)} \qquad \text{(Dupire's Formula)}$$

In the following sections, we provide a complete proof of this formula. The proof relies on the two preliminary results

- Closed-form formula for risk-neutral distribution using call option prices. See (Breeden-Litzenberger Formula)
- Kolmogrov forward equation. See (Forward Equation)

2 Implying the risk-neutral distribution

Consider $S(t)$ to be the price of an underlying asset. With $S(0) = x$, we have that

$$c(0, T, x, K) = \tilde{\mathbb{E}}\left[e^{-rT}\left(S(T) - K\right)^+\right]$$

Let $\tilde{p}(0, T, x, y)$ be the risk-neutral density in the y variable of the distribution of $S(T)$ where $S(0) = x$. We thus have that

$$c(0, T, x, K) = e^{-rT} \int_K^{+\infty} (y - K)\tilde{p}(0, T, x, y)\mathrm{d}y$$

Risk-neutral distribution is then obtained via the following equation

$$\boxed{\tilde{p}(0, T, x, K) = e^{rT} c_{KK}(0, T, x, K)} \qquad \text{(Breeden-Litzenberger Formula)}$$

Proof

Recall the following formula from elementary calculus.

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{g(x)} f(x,t)\mathrm{d}t = f(x,g(x))\cdot g'(x) + \int_0^{g(x)} \frac{\mathrm{d}f(x,t)}{\mathrm{d}x}\mathrm{d}t.$$

Note that

$$\int_K^{+\infty}(y-K)\tilde{p}(0,T,x,y)\mathrm{d}y = \int_0^{+\infty}(y-K)\tilde{p}(0,T,x,y)\mathrm{d}y - \int_0^{K}(y-K)\tilde{p}(0,T,x,y)\mathrm{d}y$$

The first and second integral have derivative $-\int_0^{+\infty}\tilde{p}(0,T,x,y)\mathrm{d}y$ and 0 respectively. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}K}\int_K^{+\infty}(y-K)\tilde{p}(0,T,x,y)\mathrm{d}y = -\int_0^{+\infty}\tilde{p}(0,T,x,y)\mathrm{d}y$$

And,

$$\frac{\mathrm{d}}{\mathrm{d}K}\int_0^{K}(y-K)\tilde{p}(0,T,x,y)\mathrm{d}y = -\int_0^{K}\tilde{p}(0,T,x,y)\mathrm{d}y$$

Summing two pieces together, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}K}\int_K^{+\infty}(y-K)\tilde{p}(0,T,x,y)\mathrm{d}y = -\int_K^{+\infty}\tilde{p}(0,T,x,y)\mathrm{d}y$$

By definition, $\int_K^{+\infty}\tilde{p}(0,T,x,y)\mathrm{d}y = \tilde{\mathbb{P}}\left(S(T)>K\right)$. Thus

$$c_K(0,T,x,K) = -e^{-rT}\int_K^{+\infty}\tilde{p}(0,T,x,y)\mathrm{d}y = -e^{-rT}\tilde{\mathbb{P}}\left(S(T)>K\right)$$

To see (Breeden-Litzenberger Formula), we need to show that

$$\frac{\mathrm{d}}{\mathrm{d}K}\int_K^{+\infty}\tilde{p}(0,T,x,y)\mathrm{d}y = -\tilde{p}(0,T,x,K),$$

which also follows from the calculus identity above. Indeed,

$$\int_K^{+\infty}\tilde{p}(0,T,x,y)\mathrm{d}y = \int_0^{+\infty}\tilde{p}(0,T,x,y)\mathrm{d}y - \int_0^{K}\tilde{p}(0,T,x,y)\mathrm{d}y$$

The first integral is constant and hence has derivative zero. The second integral's derivative is equal to $\tilde{p}(0,T,x,K)$.

3 Kolmogrov's forward and backward equation

Kolmogrov forward and backward equations were both published by Andrey Kolmogrov in 1931. Physicists were already aware of the forward equation under the name of Fokker-Plank equation. The idea behind these two equations is summerized as below.

Let $s>t$. We have information on the state of random variable x at time t(resp. s) *i.e.,* $p_t(x)$(resp. $p_s(x)$) and need to understand $p_s(x)$(resp. $p_t(x)$) in forward and backward eqs resp.

3.1 Backward equation

Consider the following SDE

$$\mathrm{d}X(u) = \beta(u, X(u))\mathrm{d}u + \gamma(u, X(u))\mathrm{d}W(u) \text{ where } X(t) = x.$$

Denote by $p(t, T, x, y)$ the transition density for the solution to this equation. In other words,

$$g(t, x) = \mathbb{E}^{t,x} h(X(T)) = \int_0^{+\infty} h(y) p(t, T, x, y)\mathrm{d}y.$$

We assume that $p(t, T, x, y) = 0$ for $0 \leq t < T$ and $y \leq 0$. Backward equation is as follows:

$$\boxed{-p_t(t, T, x, y) = \beta(t, x)p_x(t, T, x, y) + \tfrac{1}{2}\gamma^2(t, x)p_{xx}(t, T, x, y)} \qquad \text{(Backward Equation)}$$

Proof

Recall that the following PDE holds

$$g_t(t, x) + \beta(t, x)g_x(t, x) + \tfrac{1}{2}\gamma^2(t, x)g_{xx}(t, x) = 0$$

We therefore have that

$$\int_0^{+\infty} h(y) \left[p_t(t, T, x, y) + \beta(t, x)p_x(t, T, x, y) + \tfrac{1}{2}\gamma^2(t, x)p_{xx}(t, T, x, y)\right] \mathrm{d}y = 0.$$

Since this equation holds for any Borel-measurable function $h(y)$, we must have that

$$p_t(t, T, x, y) + \beta(t, x)p_x(t, T, x, y) + \tfrac{1}{2}\gamma^2(t, x)p_{xx}(t, T, x, y) = 0.$$

3.2 Forward equation

Fix x and t. Under the same assumption as in the backward case, the forward equation is as follows

$$\boxed{\frac{\partial}{\partial T}p(t, T, x, y) = -\frac{\partial}{\partial y}\left(\beta(t, y)p(t, T, x, y)\right) + \tfrac{1}{2}\frac{\partial^2}{\partial y^2}\left(\gamma^2(T, y)p(t, T, x, y)\right)} \qquad \text{(Forward Equation)}$$

Proof

Denote by

$$M(t, T, x, y) = \frac{\partial}{\partial T}p(t, T, x, y) + \frac{\partial}{\partial y}\left(\beta(t, y)p(t, T, x, y)\right) - \tfrac{1}{2}\frac{\partial^2}{\partial y^2}\left(\gamma^2(T, y)p(t, T, x, y)\right)$$

By assumption $M(t, T, x, y)$ is a continuous function of y. Therefore, for fixed t and T, if $M \not\equiv 0$, then there exists $0 < a < b$ such that $M(t, T, x, y)$ is strictly positive or strictly negative for $y \in (a, b)$. Consequently, if h is a smooth function such that

- $h(y) = 0$ for all $y \notin (a, b)$
- $h'(0) = h'(b) = 0$

- $h(y) > 0$ for all $y \in (a, b)$,

then it must hold that

$$\int_0^b h(y)M(t, T, x, y)\mathrm{d}y \neq 0.$$

It is not difficult to verify that such function h exists. For example, define

$$\alpha(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

It is easy to check that α is a smooth function. Then $\alpha(1-x)\alpha(1+x)$ is a smooth function which is identically zero outside $(-1, 1)$ and positive inside $(-1, 1)$. Itô formula gives

$$\begin{aligned} \mathrm{d}h(X(u)) &= h'(X(u))\mathrm{d}X(u) + \tfrac{1}{2}h''(X(u))\mathrm{d}X(u)\mathrm{d}X(u) \\ &= h'(X(u))\beta(u, X(u))\mathrm{d}u + h'(X(u))\gamma(u, X(u))\mathrm{d}W(u) + \tfrac{1}{2}h''(X(u))\gamma^2(u, X(u))\mathrm{d}u \end{aligned}$$

Integrating both sides from t to T gives

$$h(X(T)) = h(x) + \int_t^T \left(h'(X(u))\beta(u, X(u)) + \tfrac{1}{2}h''(X(u))\gamma^2(u, X(u))\right)\mathrm{d}u + \text{ Itô integral}$$

Since $X(u)$ has density $p(t, u, x, y)$ in the y-variable, taking expectation from both sides gives

$$\int_0^b h(y)p(t, T, x, y)\mathrm{d}y = h(x) + \int_t^T \int_0^b \left(h'(y)\beta(u, y)p(t, u, x, y) + \tfrac{1}{2}h''(y)\gamma^2(u, y)p(t, u, x, y)\right)\mathrm{d}y\mathrm{d}u.$$

On the other hand, integration by parts gives

$$\int_0^b h'(y)\beta(u, y)p(t, u, x, y)\mathrm{d}y + \int_0^b h(y)\frac{\partial}{\partial y}[\beta(u, y)p(t, u, x, y)]\mathrm{d}y = h(y)\beta(u, y)p(t, u, x, y)|_0^b = 0.$$

Here we used the fact that $h(0) = h(b) = 0$. Similarly since $h'(0) = h'(b) = 0$, we have that

$$\int_0^b h''(y)\gamma^2(u, y)p(t, u, x, y)\mathrm{d}y + \int_0^b h'(y)\frac{\partial}{\partial y}[\gamma^2(u, y)p(t, u, x, y)]\mathrm{d}y = h'(y)\gamma^2(u, y)p(t, u, x, y)|_0^b = 0.$$

Another integration by parts gives

$$\int_0^b h'(y)\frac{\partial}{\partial y}[\gamma^2(u, y)p(t, u, x, y)]\mathrm{d}y + \int_0^b h(y)\frac{\partial^2}{\partial y^2}[\gamma^2(u, y)p(t, u, x, y)]\mathrm{d}y = h(y)\frac{\partial}{\partial y}[\gamma^2(u, y)p(t, u, x, y)]|_0^b = 0.$$

We thus have shown that

$$\int_0^b h''(y)\gamma^2(u, y)p(t, u, x, y)\mathrm{d}y = \int_0^b h(y)\frac{\partial^2}{\partial y^2}[\gamma^2(u, y)p(t, u, x, y)]\mathrm{d}y.$$

Thus, we have that

$$\int_0^b h(y)p(t, T, x, y)\mathrm{d}y = h(x) + \int_t^T \int_0^b h(y)\left(-\frac{\partial}{\partial y}[\beta(u, y)p(t, u, x, y)] + \frac{\partial^2}{\partial y^2}[\gamma^2(u, y)p(t, u, x, y)]\right)\mathrm{d}y\mathrm{d}u.$$

From elementary calculus $\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{g(x)} f(t)\mathrm{d}t = f(g(x))) \cdot g'(x)$. Differentiate both sides w.r.t T to obtain

$$\int_0^b h(y)\frac{\partial}{\partial T}p(t,T,x,y)\mathrm{d}y = \int_0^b h(y)\left(-\frac{\partial}{\partial y}[\beta(T,y)p(t,T,x,y)] + \frac{\partial^2}{\partial y^2}[\gamma^2(T,y)p(t,T,x,y)]\right)\mathrm{d}y$$

Rearranging gives

$$\int_0^b h(y)M(t,T,x,y)\mathrm{d}y = 0.$$

This is the desired contradiction. As h is positive on (a,b) and zero elsewhere; But, by assumption, $M(t,T,x,y)$ is strictly positive for every $y \in (a,b)$ or strictly negative for every $y \in (a,b)$.

4 Dupire's formula

We show (Dupire's Formula) under the following assumptions for tail of $\tilde{p}(t,T,x,y)$

- $\lim_{y\to\infty}(y-K)ry\tilde{p}(0,T,x,y) = 0$
- $\lim_{y\to\infty}(y-K)\frac{\partial}{\partial y}[\sigma^2(T,y)y^2\tilde{p}(0,T,x,y)] = 0$
- $\lim_{y\to\infty}\sigma^2(T,y)y^2\tilde{p}(0,T,x,y) = 0$

Proof

Derive w.r.t. T to obtain

$$c_T(0,T,x,K) = -rc(0,T,x,K) + e^{-rT}\int_K^{+\infty}(y-K)\frac{\partial}{\partial T}\tilde{p}(0,T,x,y)\mathrm{d}y.$$

Using (Forward Equation), we have that

$$\begin{aligned} c_T(0,T,x,K) &= -rc(0,T,x,K) \\ &\quad + e^{-rT}\int_K^{+\infty}(y-K)\left[-\frac{\partial}{\partial y}\left(ry\tilde{p}(0,T,x,y)\right) + \tfrac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(T,y)y^2\tilde{p}(0,T,x,y)\right)\right]\mathrm{d}y. \end{aligned}$$

Integration by parts implies that

$$\int_K^{+\infty}(y-K)\frac{\partial}{\partial y}\left(ry\tilde{p}(0,T,x,y)\right)\mathrm{d}y + \int_K^{+\infty} ry\tilde{p}(0,T,x,y)\mathrm{d}y = (y-K)ry\tilde{p}(0,T,x,y)|_K^{+\infty}$$

By first regularity condition, right hand side is zero. Therefore,

$$-\int_K^{+\infty}(y-K)\frac{\partial}{\partial y}\left(ry\tilde{p}(0,T,x,y)\right)\mathrm{d}y = \int_K^{+\infty} ry\tilde{p}(0,T,x,y)\mathrm{d}y$$

Integration by parts and using the second regularity condition above gives

$$\begin{aligned} \int_K^{+\infty}(y-K)\frac{\partial^2}{\partial y^2}\left(\sigma^2(T,y)y^2\tilde{p}(0,T,x,y)\right)\mathrm{d}y + \int_K^{+\infty}\frac{\partial}{\partial y}\left(\sigma^2(T,y)y^2\tilde{p}(0,T,x,y)\right)\mathrm{d}y \\ = (y-K)\frac{\partial}{\partial y}\left(\sigma^2(T,y)y^2\tilde{p}(0,T,x,y)\right)|_K^{+\infty} \\ = 0. \end{aligned}$$

Therefore,

$$\int_K^{+\infty} (y-K)\frac{\partial^2}{\partial y^2}\left(\sigma^2(T,y)y^2\tilde{p}(0,T,x,y)\right)\mathrm{d}y = -\int_K^{+\infty} \frac{\partial}{\partial y}\left(\sigma^2(T,y)y^2\tilde{p}(0,T,x,y)\right)\mathrm{d}y$$
$$= \sigma^2(T,K)K^2\tilde{p}(0,T,x,K)$$

Here we used the last regularity condition. We next have that

$$\begin{aligned}
& -rc(0,T,x,K) - e^{-rT}\int_K^{+\infty}(y-K)\frac{\partial}{\partial y}\left(ry\tilde{p}(0,T,x,y)\right) \\
&= -re^{-rT}\int_K^{+\infty}(y-K)\tilde{p}(0,T,x,y)\mathrm{d}y + e^{-rT}\int_K^{+\infty} ry\tilde{p}(0,T,x,y)\mathrm{d}y \\
&= rKe^{-rT}\int_K^{+\infty}\tilde{p}(0,T,x,y)\mathrm{d}y \\
&= -rKc_K(0,T,x,K)
\end{aligned}$$

Last equality is driven from (Breeden-Litzenberger Formula). Putting pieces together, we obtain that

$$\begin{aligned}
c_T(0,T,x,K) &= -rKc_K(0,T,x,K) + \tfrac{1}{2}e^{-rT}\sigma^2(T,K)K^2\tilde{p}(0,T,x,K) \\
&= -rKc_K(0,T,x,K) + \tfrac{1}{2}\sigma^2(T,K)K^2c_{KK}(0,T,x,K).
\end{aligned}$$