Learning Quantitative Finance

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I collect my notes here as I continue with my self-studies in quantitative finance.

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Contents

0 Probability Theory Basics

Martingale: $\mathbb{E}[M(t)|\mathcal{F}(s)] = M(s)$ for all $0 \le s \le t$

Super-Martingale: $\mathbb{E}[M(t)|\mathcal{F}(s)] \le M(s)$ for all $0 \le s \le t$

Sub-Martingale: $\mathbb{E}[M(t)|\mathcal{F}(s)] \ge M(s)$ for all $0 \le s \le t$

Markov Property: Estimate of $f(X(t))$ made at time s depends only on the process value $X(s)$, *i.e.*,

$$\mathbb{E}[f(X(t))|\mathcal{F}(s)] = g(X(s))$$

Independence Lemma: If X is $\mathcal{G}$-measurable random variable and Y is independent of $\mathcal{G}$, then

$$\mathbb{E}[f(X,Y)|\mathcal{G}] = g(X) \text{ where } g(x) := \mathbb{E}f(x,Y).$$

Take out What is Known: From *Independence Lemma*, it follows that:

$$X \text{ is } \mathcal{G}\text{-measurable} \Rightarrow \mathbb{E}[XY|\mathcal{G}] = X\mathbb{E}[Y|\mathcal{G}].$$

Independence: Set $Y = 1$ in *Take out What is Known*: if X is integrable and independent of $\mathcal{G}$, then $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}X$.

Iterated Conditioning: $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]$ if $\mathcal{H} \subseteq \mathcal{G}$.

Measurability: $\mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G}$-measurable.

Partial Averaging: $Z = \mathbb{E}[X|\mathcal{G}]$ satisfies the following

$$\int_A Z d\mathbb{P} = \int_A X d\mathbb{P}, \quad \forall A \in \mathcal{G}.$$

Vice versa, any Z satisfies the preceding set of equations must be equal to $\mathbb{E}[X|\mathcal{G}]$.

1 Risk-Neutral Pricing

First fundamental theorem of asset pricing If a market model has a risk-neutral probability measure, then it does not admit arbitrage.

Second fundamental theorem of asset pricing Consider a market model that has a risk-neutral probability measure. The model is complete if and only if the risk-neutral probability measure is unique.

1.1 Forwards and Futures

T-forward price at time t Value of K that makes the forward contract's no-arbitrage price 0 at time t

Theorem: $K = \frac{S(t)}{B(t,T)}$

Proof (No-arbitrage)

- At time t:
  - Sell forward contract
  - Short $\frac{S(t)}{B(t,T)}$ zero-coupon bonds
  - Buy one share of stock
- At time T:
  - Give stock & receive K
  - Pay $\frac{S(t)}{B(t,T)}$ to cover zero-coupon bond

  Payoff is equal $K - \frac{S(t)}{B(t,T)}$.

Default risk with forward contracts Suppose $t_1 < t_2$ and the agent enters into a forward contract at t_1. Value of this contract at t_2 is

$$\begin{aligned}\frac{1}{D(t_2)}\tilde{\mathbb{E}}\left[D(T)\left(S(T) - \frac{S(t_1)}{B(t_1,T)}\right)|\mathcal{F}(t_2)\right] &= S(t_2) - \frac{S(t_1)}{B(t_1,T)} \cdot \frac{1}{D(t_2)}\tilde{\mathbb{E}}\left[D(T)|\mathcal{F}(t_2)\right] \\ &= S(t_2) - S(t_1) \cdot \frac{B(t_2,T)}{B(t_1,T)}\end{aligned}$$

If interest rate is constant r, then

$$S(t_2) - S(t_1) \cdot \frac{B(t_2,T)}{B(t_1,T)} = S(t_2) - S(t_1) \cdot e^{r(t_2 - t_1)}$$

If asset price grows faster than the interest rate, then forward contract gains value. Otherwise, its value becomes negative. This raises the risk of default. One potential solution is to open and close forward contracts frequently. Lack of liquidity renders this idea useless. Moreover, this solution does not provide any hedging utility.

Future prices If the agent holds future position between t_k and t_{k+1}, then at time t_{k+1}, he would receive a payment

$$\mathrm{Fut}_S(t_{k+1}, T) - \mathrm{Fut}_S(t_k, T)$$

This is called *marking to margin*. The following two key points must hold:

- $\mathrm{Fut}_S(T, T) = S(T)$
- At each time t_k, the value of the payment to be received at t_{k+1} is zero.

From the second item, we have that

$$0 = \frac{1}{D(t_k)} \tilde{\mathbb{E}} \left[D(t_{k+1}) \left(\mathrm{Fut}_S(t_{k+1}, T) - \mathrm{Fut}_S(t_k, T) \right) | \mathcal{F}(t_k) \right]$$

We assume the interest rate is constant within two consecutive time stamps. Thus

$$\begin{aligned} D(t_{k+1}) &= \exp\left(- \int_0^{t_{k+1}} R(u) \mathrm{d}u \right) \\ &= \exp\left(- \sum_{j=0}^{k} R(t_j)(t_{j+1} - t_j) \right) \in \mathcal{F}(t_k) \end{aligned}$$

Therefore,

$$\tilde{\mathbb{E}} \left[\mathrm{Fut}_S(t_{k+1}, T) | \mathcal{F}(t_k) \right] = \mathrm{Fut}_S(t_k, T)$$

Therefore, using bullet point I, we have

$$\mathrm{Fut}_S(t_k, T) = \tilde{\mathbb{E}} \left[S(T) | \mathcal{F}(t_k) \right]$$

Theorem Value of a long(or a short) position to be held between an interval of time is 0.

Valuation of a cash flow Consider an asset that pays $C(u) \in \mathcal{F}(u)$. Holding one share of this asset, produces the following differential

$$\mathrm{d}X(u) = \mathrm{d}C(u) + R(u)X(u)\mathrm{d}u.$$

If $X(t) = 0$, then the value of the cash flow received between t and T is

$$\frac{1}{D(t)} \tilde{\mathbb{E}} \left[\int_t^T D(u) \mathrm{d}C(u) | \mathcal{F}(t) \right], \quad 0 \leq t \leq T.$$

Future-Forward spread If interest rate is constant then $\mathrm{Fut}_S(0, T) = \mathrm{For}_S(0, S)$. However,

$$\begin{aligned} \underset{S}{\mathrm{For}}(0, S) - \mathrm{Fut}_S(0, T) &= \frac{S(0)}{B(0, T)} - \tilde{\mathbb{E}}[S(T)] \\ &= \frac{\tilde{\mathbb{E}}[D(T)S(T)]}{B(0, T)} - \tilde{\mathbb{E}}[S(T)] \\ &= \frac{1}{B(0, T)} \left[\tilde{\mathbb{E}}[D(T)S(T)] - \tilde{\mathbb{E}}[D(T)]\tilde{\mathbb{E}}[S(T)] \right] \\ &= \frac{1}{B(0, T)} \widetilde{\mathrm{Cov}} \left(D(T), S(T) \right) \end{aligned}$$

2 Greeks

Consider price of an option $P(t, S, \sigma, r)$. Here S, r and σ indicates stock price, interest rate and implied volatility resp. Second order Taylor expansion of P is as follows.

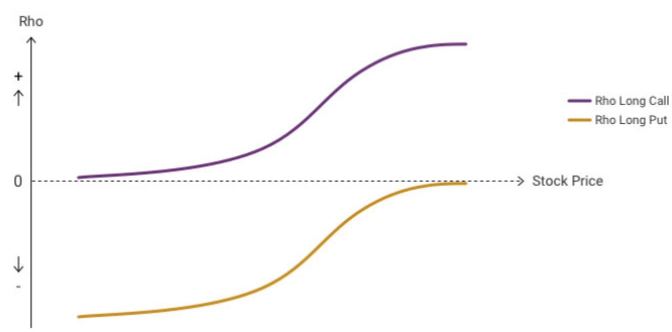
$$\mathrm{d}P = \underbrace{\frac{\mathrm{d}P}{\mathrm{d}t}}_{:=\Theta}\mathrm{d}t + \underbrace{\frac{\mathrm{d}P}{\mathrm{d}S}}_{:=\Delta}\mathrm{d}S + \underbrace{\frac{\mathrm{d}P}{\mathrm{d}\sigma}}_{:=v}\mathrm{d}\sigma + \underbrace{\frac{\mathrm{d}P}{\mathrm{d}r}}_{:=\rho}\mathrm{d}r + \frac{1}{2}\underbrace{\frac{\mathrm{d}^2P}{\mathrm{d}S^2}}_{:=\Gamma}\mathrm{d}S\mathrm{d}S + \frac{1}{2}\underbrace{\frac{\mathrm{d}^2P}{\mathrm{d}S\mathrm{d}\sigma}}_{:=\text{Vanna}}\mathrm{d}S\mathrm{d}\sigma + \frac{1}{2}\underbrace{\frac{\mathrm{d}^2P}{\mathrm{d}\sigma^2}}_{:=\text{Volga}}\mathrm{d}\sigma\mathrm{d}\sigma + \frac{1}{2}\underbrace{\frac{\mathrm{d}^2P}{\mathrm{d}S\mathrm{d}t}}_{:=\text{Charm}}\mathrm{d}S\mathrm{d}t + \text{re}$$

Moneyness & Δ $\Delta_C \in [0, 1]$ and $\Delta_P \in [-1, 0]$. + for long positions and - for short positions.

- **ATM Call** $\Delta \approx 0.5$
- **ITM Call** $\Delta \gtrapprox 0.5$
- **OTM Call** $\Delta \lessapprox 0.5$

- **ATM Put** $\Delta \approx -0.5$
- **ITM Put** $\Delta \lessapprox -0.5$
- **OTM Put** $\Delta \gtrapprox 0.5$

ρ **Sign** Long options (long call & short put) have + ρ. Short options have - ρ. Indeed, ρ is + for purchased calls as higher interest rates increase call premiums.



3 Connections with Partial Differential Equations

Two ways to price a derivative security

- Monte Carlo Simulation
- Numerically solve a PDE

Question: How to connect risk-neutral pricing with partial differential equations?

Stochastic differential equations (SDE) are used to model asset prices.

$$\mathrm{d}X(u) = \underbrace{\beta(u, X(u))}_{\text{drift}}\,\mathrm{d}u + \underbrace{\gamma(u, X(u))}_{\text{diffusion}}\,\mathrm{d}W(u) \text{ with } \underbrace{X(t) = x}_{\text{initial condition}} \tag{SDE}$$

Theorem (Markov property): Solutions to SDE have Markov property.

Example (One-dimensional linear SDE)

$$\mathrm{d}X(u) = (a(u) + b(u)X(u))\,\mathrm{d}u + (\gamma(u) + \sigma(u)X(u))\,\mathrm{d}W(u) \text{ with } X(t) = x.$$

Solutions to this SDE have closed-form formula. See Exercise 6.1. It is emphasized that to ensure solutions' Markov property, $a(u), b(u), \gamma(u)$, and $\sigma(u)$ are non-random.

Example (Geometric Brownian Motion):

$$\beta(u, x) = \alpha x, \gamma(u, x) = \sigma x.$$

$$\mathrm{d}S(u) = \alpha S(u)\mathrm{d}u + \sigma S(u)\mathrm{d}W(u)$$

$$S(T) = x \cdot e^{\sigma(W(T)-W(t)) + \left(\alpha - \frac{1}{2}\sigma^2\right)(T-t)}$$

Interest rate models

$$\mathrm{d}R(u) = \beta(u, R(u))\mathrm{d}u + \gamma(u, R(u))\mathrm{d}\tilde{W}(u) \tag{Interest Rate SDE}$$

Example (Hull-White interest rate model)

$$\mathrm{d}R(u) = (a(u) - b(u)R(u))\,\mathrm{d}u + \sigma(u)\mathrm{d}\tilde{W}(u)$$

This SDE has closed-form formula. In particular, $R(T)$ is normally distributed and hence $R(T) < 0$ with positive probability; a draw-back of HW model.

Example (CIR interest rate model)

$$\mathrm{d}R(u) = (a - bR(u))\,\mathrm{d}u + \sigma\sqrt{R(u)}\mathrm{d}\tilde{W}(u)$$

MGF of $R(u)$ is obtained using Ornstein-Uhlenbeck SDE:

$$\mathrm{d}X_j(t) = -\frac{b}{2}X_j(t)\mathrm{d}t + \frac{1}{2}\sigma\mathrm{d}W_j(t).$$

Letting $R(t) = \sum_{j=1}^{d} X_j^2(t)$ and $B(t) = \sum_{j=1}^{d}\int_0^t \frac{X_j(s)}{\sqrt{R(s)}}\mathrm{d}W_j(s)$. Then

$$\mathrm{d}R(u) = (a - bR(u))\,\mathrm{d}u + \sigma\sqrt{R(u)}\mathrm{d}B(u)$$

Let X be a solution for (SDE). Denote

$$g(t,x) := \mathbb{E}^{t,x}h(X(T))$$

Here the initial condition is $X(t) = x$.

Theorem (Markov Property) With initial condition $X(0) = x_0$, it holds

$$\mathbb{E}^{t,x}h(X(T)) = \mathbb{E}^{0,x_0}\left[h(X(T))|X(t) = x\right]$$

Feynman-Kac Connects an SDE with a PDE.

$$g_t(t,x) + \beta(t,x)g_x(t,x) + \frac{1}{2}\gamma^2(t,x)g_{xx}(t,x) = 0 \text{ where } g(T,x) = h(x), \forall x.$$

Denote

$$f(t,x) = \mathbb{E}\left[e^{-r(T-t)}h(X(T))\right]$$

Discounted Feynman-Kac

$$f_t(t,x) + \beta(t,x)f_x(t,x) + \frac{1}{2}\gamma^2(t,x)f_{xx}(t,x) = rf(t,x) \text{ where } f(T,x) = h(x), \forall x.$$

Example (Options on Geometric Brownian Motion)

- $h(S(T))$: Payoff at time T
- Underlying asset: $\mathrm{d}S(u) = rS(u)\mathrm{d}u + \sigma S(u)\mathrm{d}\tilde{W}(u)$
- Price at time t: $v(t) = \tilde{\mathbb{E}}\left[e^{-r(T-t)}h(S(T))|\mathcal{F}(t)\right]$

Discounted Feynman-Kac produces BSM again.

$$v_t(t,x) + rxv_x(t,x) + \frac{1}{2}\sigma^2x^2v_{xx}(t,x) = rv(t,x).$$

Here σ could be random and depend on t, x.

Discount process

$$D(t) = e^{-\int_0^t R(s)\mathrm{d}s}$$

Money market account price process

$$M(t) = \frac{1}{D(t)} = e^{\int_0^t R(s)\mathrm{d}s}$$

Bond prices The following holds

$$\begin{aligned} D(t)B(t,T) &= \tilde{\mathbb{E}}\left[D(T)|\mathcal{F}(t)\right] \\ B(t,T) &= \tilde{\mathbb{E}}\left[e^{-\int_t^T R(s)\mathrm{d}s}|\mathcal{F}(t)\right] \\ f(t,R(t)) := B(t,T) &= e^{-Y(t,T)(T-t)} \end{aligned}$$

Yield $Y(t,T)$ yield between time t and T

R satisfies (Interest Rate SDE), $D(t)B(t,T)$ is Markov. Discounted Feynman-Kac gives

$$f_t(t,r) + \beta(t,r)f_r(t,r) + \frac{1}{2}\gamma^2(t,r)f_{rr}(t,r) = rf(t,r) \text{ where } f(T,r) = 1, \forall r.$$

Option on bonds $0 \le t \le T_1 \le T_2$. A call option with expiry T_1 to buy a bond with expiry at T_2. The following is true.

$$c(t,R(t)) = \tilde{\mathbb{E}}\left[e^{-\int_t^T R(s)\mathrm{d}s} \cdot (f(T_1,R(T_1)) - k)^+\right]$$

The following PDE holds

$$c_t(t,r) + \beta(t,r)c_r(t,r) + \frac{1}{2}\gamma^2(t,r)c_{rr}(t,r) = rc(t,r) \text{ where } c(T_1,r) = (f(T_1,r) - K)^+, \forall r.$$

This is the same PDE as for bond prices only with different terminal conditions.

Multidimensional Feynman-Kac For $i = 1, 2$

$$\mathrm{d}X_i(u) = \beta_i(u,X_1(u),X_2(u))\mathrm{d}u + \gamma_{i1}(u,X_1(u),X_2(u))\mathrm{d}W_1(u) + \gamma_{i2}(u,X_1(u),X_2(u))\mathrm{d}W_2(u).$$

Denote

$$g(t,x_1,x_2) = \mathbb{E}^{t,x_1,x_2}h(X_1(T),X_2(T)) \text{ where } X_i(t) = x_i.$$

Then

$$g_t + \beta_1 g_{x_1} + \beta_2 g_{x_2} + \frac{1}{2}\left(\gamma_{11}^2 + \gamma_{12}^2\right)g_{x_1x_1} + \frac{1}{2}\left(\gamma_{21}^2 + \gamma_{22}^2\right)g_{x_2x_2} + \left(\gamma_{11}\gamma_{21} + \gamma_{12}\gamma_{22}\right)g_{x_1x_2} = 0.$$

Example (Asian option) Asian options are path-independent. Their pay-off is

$$v(T) = \left(\frac{1}{T} \int_0^T S(u) \mathrm{d}u - K \right)^+$$

Denote

$$Y(t) := \int_0^t S(u) \mathrm{d}u \Rightarrow \mathrm{d}Y(u) = S(u) \mathrm{d}u.$$

For $0 \leq t \leq T$

$$v(t, S(t), Y(t)) = \tilde{\mathbb{E}} \left[e^{-r(T-t)} \left(\frac{1}{T} Y(T) - K \right)^+ | \mathcal{F}(t) \right]$$

$Y(u)$ is not Markov, but $(S(u), Y(u))$ is. Multi-dimensional Feynman-Kac gives

$$v_t(t,x,y) + rxv_x(t,x,y) + \underbrace{xv_y(t,x,y)}_{\text{new term}} + \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x,y) = rv(t,x,y).$$

Termination condition is

$$v(T,x,y) = \left(\frac{y}{T} - K \right)^+.$$

Kolmogorov backward and forward equations Consider the following SDE

$$\mathrm{d}X(u) = \beta(u, X(u))\mathrm{d}u + \gamma(u, X(u))\mathrm{d}W(u) \text{ where } X(t) = x.$$

Denote by $p(t,T,x,y)$ the transition probability. Backward equations fix forward variables *i.e.,* T and y and derive w.r.t backward variables *i.e.,* t and x:

$$-p_t(t,T,x,y) = \beta(t,x)p_x(t,T,x,y) + \tfrac{1}{2}\gamma^2(t,x)p_{xx}(t,T,x,y)$$

Forward equation do the reverse:

$$\frac{\partial}{\partial T} p(t,T,x,y) = -\frac{\partial}{\partial y}\left(\beta(t,y)p(t,T,x,y) \right) + \tfrac{1}{2} \frac{\partial^2}{\partial y^2} \left(\gamma^2(T,y)p(t,T,x,y) \right)$$

4 Exotic Options

Exotic Options whose payoffs depend on path of the underlying asset are called path-dependent

Example Three types of exotic options on geometric Brownian motions assets are considered:

- Barrier (*e.g.,* up & out) options - Explicit pricing formula ✓
- Lookback options - Explicit pricing formula ✓
- Asian options - Explicit pricing formula X - Numerical friendly PDE ✓

Reflection Principle The first two options are priced analytically using reflection principle.

$$\mathbb{P}\left(\tau_m \leq t, W(t) \leq \omega\right) = \mathbb{P}\left(W(t) \leq 2m - \omega\right) \quad \omega \leq m, m > 0.$$

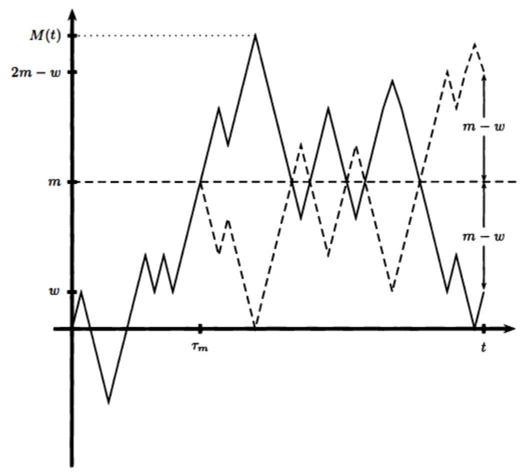


Fig. 3.7.1. Brownian path and reflected path.

Using reflection principle, we can compute the joint density of $M(t)$ and $W(t)$

$$\mathbb{P}\left(M(t) \geq m, W(t) \leq \omega\right) = \mathbb{P}\left(W(t) \geq 2m - \omega\right), \quad w \leq m, m \geq 0.$$

From $S(t) = rS(t)\mathrm{d}t + \sigma S(t)\mathrm{d}\widetilde{W}(t)$, obtain the following

$$S(t) = S(0)e^{\sigma\widehat{w}(t)}, \quad \alpha = \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right)$$

Applying a change of measure argument, we could compute joint density of $\widehat{M}(t)$ and $\widehat{W}(t)$.

$$\widehat{W}(t) = \alpha t + W(t), \quad \widehat{M}(t) = \max_{0 \leq s \leq t} \widehat{W}(s).$$

Payoffs Payoff functions for barriers and lookbacks are computed as follows:

$$V_{\text{barrier}}(T) = \left(S(0)e^{\sigma\widehat{W}(t)} - K\right)\mathbf{1}_{\{\widehat{W}(T)\geq k, \widehat{M}(T)\leq b\}}$$
$$V_{\text{lookback}}(T) = S(0)\left(e^{\sigma\widehat{M}(T)} - e^{\sigma\widehat{W}(T)}\right)$$

Boundary conditions

$$\begin{aligned}
\mathcal{R}_{\text{Barrier}} &= \{(t,x) : 0 \le t < T, 0 \le x \le B\} \\
v^B(t,0) &= 0, 0 \le t \le T \\
v^B(t,B) &= 0, 0 \le t < T \\
v^B(T,x) &= (x-K)^+, 0 \le x \le B
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}_{\text{Lookback}} &= \{(t,x,y) : 0 \le t < T, 0 \le x \le y\} \\
v^L(t,0,y) &= e^{-r(T-t)}y, 0 \le t \le T, y \ge 0 \\
v^L_y(t,y,y) &= 0, 0 \le t < T, y > 0 \\
v^L(T,x,y) &= y - x, 0 \le x \le y
\end{aligned}$$

Partial Differential Equations The call has not been knocked out by t and $S(t) = x$

$$v^B_t(t,x) + rxv^B_x(t,x) + \frac{1}{2}\sigma^2 x^2 v^B_{xx}(t,x) = rv^B(t,x) \qquad \forall (t,x) \in \mathcal{R}_{\text{Barrier}}.$$

$$v^L_t(t,x,y) + rxv^L_x(t,x,y) + \frac{1}{2}\sigma^2 x^2 v^L_{xx}(t,x,y) = rv^L(t,x,y) \qquad \forall (t,x,y) \in \mathcal{R}_{\text{Lookback}}.$$

Delta-hedging for Barriers $v(t,x)$ is discontinuous at the corner of $\mathcal{R}_{\text{Barrier}}$ at which delta (*i.e.,* $v_x(t,S(t))$) and gamma (*i.e.,* $v_{xx}(t,S(t))$) are large negative values. Normal delta-hedging becomes impractical as the large volume of trades renders significant the presumably negligible bid-ask spread.The common industry practice is to price and hedge the up-and-out call as if the barrier were at a level slightly higher than B.

dY(t) $\mathrm{d}Y(t)$ is different from $\mathrm{d}S(t)$ and $\mathrm{d}t$. This follows from the fact that $Y(t)$ is monotonic and thus has zero quadratic variation. Moreover, $Y(t)$'s flat regions has Lebesgue measure 1 and hence $\mathrm{d}Y(t) \ne \Theta(t)\mathrm{d}t$ for any procecss $\Theta(t)$. The following holds

$$\begin{aligned}
\mathrm{d}Y(t)\mathrm{d}Y(t) &= 0 \\
\mathrm{d}Y(t)\mathrm{d}S(t) &= 0 \\
\mathrm{d}e^{-rt}v(t,S(t),Y(t)) &= e^{-rt}[\cdots]\mathrm{d}t + e^{-rt}\sigma S(t) v_x(t,S(t),Y(t))\mathrm{d}\widetilde{W}(t) + e^{-rt}v_y(t,S(t),Y(t))\mathrm{d}Y(t) \\
\mathrm{d}Y(t) \ne 0 &\text{ iff } S(t) = Y(t) \Rightarrow \text{ 2nd boundary condition}
\end{aligned}$$

4.1 Exotic Options Summary

- **Binary(Digital) (2)**
  - Cash-Or-Nothing: Payoff $= X$ if $S(T) > K$ else 0
  - Asset-Or-Nothing: Payoff $= S(T)$ if $S(T) > K$ else 0
- **Barrier (4)**

  - Up-and-Out Or Up-and-In
  - Down-and-Out Or Down-and-In

- **Asian (2)**
  - Fixed-Strike (average price used in place of asset price): Payoff 4Call $= \max(A_T - K, 0)$
  - Floating-Strike (average price used in place of strike): Payoff 4Call $= \max(S_T - A_T, 0)$
- **Lookback (2)**
  - Fixed-Strike: Payoff 4Call $= \max(S_{\max} - K, 0)$
  - Floating-Strike: Payoff 4Call $= \max(S(T) - S_{\min}, 0)$
- **Bermuda** There are predetermined exercise dates before expiry. $t_1, \cdots, t_n = T$.
- **Spread** Payoff 4Call: $\max\{(S_1(T) - S_2(T)) - K, 0\}$, 4Put: $\max\{K - (S_1(T) - S_2(T)), 0\}$
- **Range** Payoff 4Call: $\max\{(\max(T) - \min(T)) - K, 0\}$, 4Put: $\max\{K - (\max(T) - \min(T)), 0\}$
- **Basket** Payoff 4Call: $\max\{\sum_{i=1}^{N} w_i S_i(T) - K, 0)\}$
- **Chooser** At a predetermined date, owner decides between Put or Call
- **Compound** Ex: Call of Call. Payoff $= \max\{C(S_{T_c}, K, T - T_c) - K_c, 0\}$ $(T_c < T)$
- **Worst of or Best of** Ex: Worst of Put. Payoff $= \max\{K - \underbrace{\min(S_1(T), \cdots, S_n(T))}_{\text{worst of part}}, 0\}$
- **Extendable** Owner has the right to extend the maturity date
- **Cliquet** Series of consecutive forward starting ATM options

5 American Derivative Securities

American option Owner can exercise at any time up to and including the expiration date

American call Early exercise for a call on a stock paying no dividend is worthless

American put *Early exercise premium* may be substantial

Bermudan Early exercise is possible at only contractually prespecified dates

Intrinsic value Payoff of an American option could not be less than the one associated with its immediate exercise. This is called the intrinsic value of the option.

Supermartingale property Discounted price of an American option is a supermartingale (*i.e.,* tend to fall) under the risk neutral measure. During the time which is not optimal to exercise, the discounted price process behaves as a martingale though

Optimal exercise time Worst time for the seller as well as the best time for the owner to exercise

Stopping time A stopping time τ is a random variable taking values in $[0, \infty]$ and satisfying

$$\{\tau \leq t\} \in \mathcal{F}(t) \text{ for all } t \geq 0.$$

Optional sampling A stopped martingale(*resp.* supermartingale, submartingale) is a martingale(*resp.* supermartingale, submartingale).

5.1 Perpetual American put

Perpetual American put Price of a perpetual American put is defined as below

$$v_*(S(0)) = \max_{\tau \in \mathcal{T}} \tilde{\mathbb{E}}\left[e^{-r\tau}\left(K - S(\tau)\right)\right]$$

Here $\mathcal{T}$ denotes the set of all stopping time.

Hedging $v_*(S(0))$ is the initial capital required to hedge a short position in the American put regardless of the exercise strategy used by the owner.

Level L_* There is no expiration and optimal exercise strategy should only depend on the value $S(t)$. We guess (and prove later) that the optimal exercise strategy is as follows.

$$\text{Exercise once } S(t) \leq L_*$$

If $S(0) < L_*$ then the owner exercises immediately and receives the intrinsic value.

Laplace transform for first passage time of drifted Brownian motion Denote $X(t) = \mu t + \tilde{W}(t)$ and set

$$\tau_m = \min\{t \geq 0 : X(t) = m\}$$

Set $\tau_m = \infty$ if $X(t)$ never reaches m. The following is true for $\lambda > 0$.

$$\tilde{\mathbb{E}} e^{-\lambda \tau_m} = e^{-m(-\mu + \sqrt{\mu^2 + 2\lambda})}$$

Letting $\lambda \downarrow 0$,

$$\tilde{\mathbb{P}}(\tau_m < +\infty) = e^{m\mu - m|\mu|}$$

Let $L < K$ and set

$$\begin{cases} \tau_L = 0 & \text{if } S(0) \leq L \\ \tau_L = \min\{t \geq 0 : S(t) = L\} & \text{if } S(0) > L \end{cases}$$

The value of the put under this strategy is computed as follows.

$$v_L(S(0)) = (K - L)\, \tilde{\mathbb{E}} e^{-r\tau_L(S(0))}$$

Value of $v_L(S(0))$ Using the Laplace transform mentioned above, we obtain that

$$v_L(S(0)) = \begin{cases} K - S(0) & 0 \leq S(0) \leq L \\ (K - L)\left(\frac{S(0)}{L}\right)^{-\frac{2r}{\sigma^2}} & S(0) \geq L. \end{cases}$$

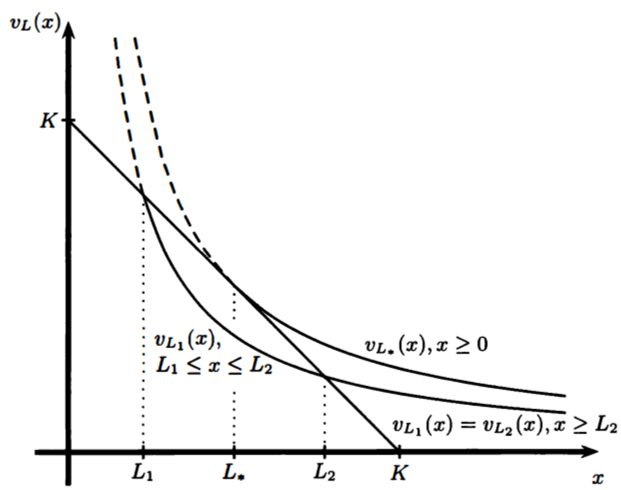


Fig. 8.3.1. $(K - L)\left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}}$ **for three values of** L.

From Figure above, it is clear that $v_L(x)$ is maximized for each fix x for L_* which satisfies the *smooth pasting* condition *i.e.,*

$$v'_{L^*}(L_*+) = v'_{L^*}(L_*-) = -1.$$

L_* is computed explicitly below.

$$L_* = \frac{2r}{2r + \sigma^2} K$$

The following is true.

$$rv_{L_*}(x) - rxv'_{L_*}(x) - \frac{1}{2}\sigma^2 x^2 v''_{L_*}(x) = \begin{cases} 0 & \text{if } x > L_* \\ rK & \text{if } 0 \le x < L_* \end{cases}$$

Linear complimentary conditions (satisfied by $v_{L^*}(x)$) are as below.

- $v(x) \ge (K - x)^+$ for all $x \ge 0$
- $rv(x) - rxv'(x) - \frac{1}{2}\sigma^2 x^2 v''(x) \ge 0$ for all $x \ge 0$
- At least one of the above two inequalities holds with equality for each $x \ge 0$

Supermartingale for $e^{-rt}v_{L^*}(S(t))$ The following is true.

$$\mathrm{d}e^{-rt}v_{L^*}(S(t)) = \underbrace{-e^{-rt}rK\mathbf{1}_{\{S(t)<L_*\}}}_{\le 0}\mathrm{d}t + e^{-rt}\sigma S(t)v'_{L_*}(S(t))\mathrm{d}\tilde{W}(t).$$

In particular,

$$\mathrm{d}e^{-r(t\wedge\tau_{L^*})}v_{L^*}(S(t\wedge\tau_{L^*})) = e^{-r(t\wedge\tau_{L^*})}\sigma S(t\wedge\tau_{L^*})v'_{L_*}(S(t\wedge\tau_{L^*}))\mathrm{d}\tilde{W}(t).$$

Optimal strategy v_{L_*} Let τ be the set of all stopping times (including $\tau = +\infty$). Then

$$v_{L_*}(x) = \max_{\tau\in\mathcal{T}} \tilde{\mathbb{E}}\left[e^{-r\tau}\left(K - S(\tau)\right)\right]$$

For $\tau = +\infty$, the argument is defined as zero. It is emphasized that + sign is omitted from the payoff function.

Cash consumption Hedging short American put options is similar to the European version during the time that the discounted asset price is a martingale. If the optimal exercise stopping time is missed, the discounted asset price is only a supermartingale and hence the agent can consume cash while maintaining the hedge. In other words,

$$\mathrm{d}X(t) = \Delta(t)\mathrm{d}S(t) + r\left(X(t) - \Delta(t)S(T)\right)\mathrm{d}t - C(t)\mathrm{d}t$$

Here

$$C(t) = rK\mathbf{1}_{\{S(t)<L_*\}}$$

In words, $X(t) = K - S(t)$ before exercise where K is invested in the money market and one share of stock is short. Should the owner exercise, the agent receives $S(t)$ and pays back K from his money market account.

Importance of linear complimentary conditions The first two conditions are required to satisfy put's seller *i.e., hedge is possible*

- $V(t) \ge (K - S(t))^+$
- $e^{-rt}V(t)$ is a supermartingale under $\tilde{\mathbb{P}}$

The last condition is needed to ensure the owner that there exists an exercise strategy which captures option's full value.

- $V(0) = \tilde{\mathbb{E}}\left[e^{-r\tau_*}\left(K - S(\tau_*)\right)^+\right]$

Continuation set The owner should not exercise while inside

$$\mathcal{C} = \{x \geq 0 : v_{L_*}(x) > (K - x)^+\}$$

Stopping set The owner should exercise once inside

$$\mathcal{S} = \{x \geq 0 : v_{L_*}(x) = (K - x)^+\}$$

5.2 Finite Expiration American Put

$\mathcal{T}_{t,T}$ a stopping time in $\mathcal{T}_{t,T}$ stops at $u \in [t, T]$ based on the path of stock price between $[t, u]$

Pricing Price of American put expires at T at time t is computed as below

$$v(t, x) = \max_{\tau \in \mathcal{T}_{t,T}} \tilde{\mathbb{E}} \left[e^{-r(\tau - t)} (K - S(\tau)) \,|S(t) = x \right]$$

Level $L(T-t)$ Due to $T < +\infty$, level L_* now depends on time to expiration. It is known that $L(\tau)$ decreases as τ increases.

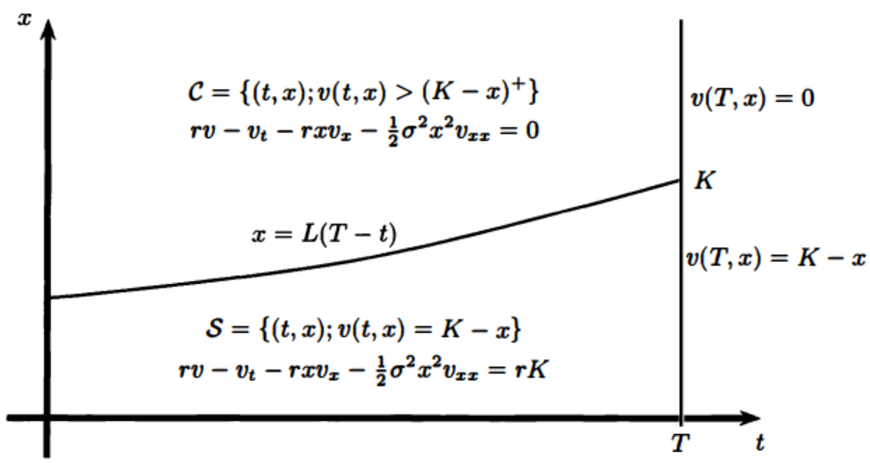


Fig. 8.4.1. Finite-expiration American put.

Linear complimentary conditions From the figure above, notice the difference inside the linear complimentary conditions. The smooth pasting condition is also as below.

$$v_x(t, x+) = v_x(t, x-) = -1 \text{ for } x = L(T-t),\ 0 \leq t < T.$$

Smooth pasting condition does not hold at T, but $L(0) = K$.

Finite difference scheme Via equations below, one solves for $v(t,x)$ and $L(T-t)$ simultaneously.

- $rv(t,x) - v_t(t,x) - rxv_x(t,x) - \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x) = 0$ for $x \geq L(T-t)$
- $v(t,x) = K - x$ for $0 \leq x \leq L(T-t)$
- $v_x(t,x+) = v_x(t,x-) = -1$ for $x = L(T-t),\ 0 \leq t < T$
- $\lim_{x\to+\infty} v(t,x) = 0$ *i.e.,* more valuable stock $\equiv$ less valuable put

5.3 American Call

Main Lemma Let $h \geq 0$ and convex. Payoff upon exercise is $h(S(t))$. The discounted intrinsic value *i.e.,* $e^{-rt}h(S(t))$ is a submartingale.

Not-paying dividend assets Early exercise for American derivative securities for these assets with payoff $h(S(T))$ is useless.

$$\underbrace{\tilde{\mathbb{E}}\left[e^{-r(T-u)}h(S(T))|\mathcal{F}(u)\right]}_{\text{European call payoff with start } S(u) \text{ expiry at } T} \quad \geq \quad \underbrace{h(S(u))}_{\text{intrinsic value at time } u}$$

In particular, American call and European call (expiry, strike held equal) have the same price. Indeed, $e^{-rt}(S(t)-K)^+$ is a submartingale and hence tend to rise.

Dividend paying assets Consider the dividend payments as below

$$0 < t_1 < t_2 < \cdots < t_n < T.$$

Optimal exercise time The only potential exercise time is right before a dividend payment.

$$\underbrace{S(t_j)}_{\text{asset price after dividend payment}} \quad = \quad \underbrace{S(t_j-)}_{\text{asset price just prior to dividend payment}} \quad - \quad \underbrace{a_j S(t_j-)}_{\text{dividend payment}}$$

The optimal exercise time is immediately prior to the dividend payment at the smallest time t_j for which $S(t_j-) - K$ exceeds $c_j(t_j, (1-a_j)S(t_j-))$.

Call value before dividend payment Right before t_n, the agent either exercises, in which case receives $S(t_n-) - K$, or declines to exercise, in which case the dividend is paid and the asset price drops to $(1-a_n)S(t_n-)$. In the latter case, the call value for $t_n < t < T$ is determined through Black-Scholes-Merton formula as below.

$$\frac{\partial}{\partial t_n}c_n(t,x) + rx\frac{\partial}{\partial x}c_n(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}c_n(t,x) = rc_n(t,x),\ t_n \leq t < T, x \geq 0.$$

Terminal condition is

$$c_n(T,x) = (x-K)^+, \quad x \geq 0.$$

Therefore, the call value at t_n is equal to $h_n(S(t_n-))$ where

$$h_n(t_n,x) = \max\{x-K, c_n(t_n,(1-a_n)x)\}, x \geq 0.$$

Convexity of $c_n(t,x)$ $c_n(t,x)$ is convex and so is $h_n(t,x)$ in x.

Why not exercise inside $[t_{n-1}, t_n-)$ This follows immediately from convexity of h_n

$$\begin{aligned} c_{n-1}(t, S(t)) := \tilde{\mathbb{E}}\left[e^{-r(t_n - t)} h_n(S(t_n -))|\mathcal{F}(t)\right] &\geq h_n(S(t)) \\ &\geq S(t) - K. \end{aligned}$$

The following terminal condition holds

$$c_{n-1}(t_n, x) = h_n(t_n, x).$$

Black-Scholes-Merton equation holds for $c_{n-1}(t,x)$ as below.

$$\frac{\partial}{\partial t_n} c_{n-1}(t,x) + rx\frac{\partial}{\partial x} c_{n-1}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} c_{n-1}(t,x) = rc_{n-1}(t,x),\ t_{n-1} \leq t < t_n, x \geq 0.$$

Recursion The following diagram illustrates the recursion used in computing the price of an American call on a dividend paying asset.

$c_n(T,x) = (x-K)^+$	Ter. cond. at T for BSM on $[t_n, T)$
$\to h_n(t_n, x) = \max\{x-K, c_n(t_n, (1-a_n)x)\}$	Payoff at dividend date t_n
$\to c_{n-1}(t_n, x) = h_n(t_n, x)$	Ter. cond. at t_n for BSM on $[t_{n-1}, t_n)$
$\to h_{n-1}(t_{n-1}, x) = \max\{x-K, c_{n-1}(t_{n-1}, (1-a_{n-1})x)\}$	Payoff at dividend date t_{n-1}
$\to c_{n-2}(t_{n-1}, x) = h_{n-1}(t_{n-1}, x)$	Ter. cond. at t_{n-1} for BSM on $[t_{n-2}, t_{n-1})$

After solving n differential equations, we have explicit formula for the functions below

$$h_1(x), \cdots, h_n(x)$$

As well as

$$c_1(t,x), \cdots, c_n(t,x)$$

The optimal time to exercise is the first t_i when

$$h_i(S(t_i-)) = S(t_i-) - K.$$

6 Change of Numeraire

Numeraire Unit of account in which assets are denominated. Example: Currency of a country

Change of numeraire Change from one currency to another. It's necessary due to 1) finance consideration 2) model consideration

Dividend Numeraire can't pay dividends. For example, currency can't be taken as numeraire as it pays dividends when invested in the money market account.

6.1 Multidimensional market model

Source of uncertainty $d-$dimensional Brownian motion: $(W_1(t), \cdots, W_d(t))$ (indep.)

Assets m primary assets

$$\mathrm{d}S_i(t) = \alpha_i(t)S_i(t)\mathrm{d}t + S_i(t)\sum_{j=1}^{d}\sigma_{i,j}(t)\mathrm{d}W_j(t)$$

Market price of risk (Assumption) $\exists$ a unique $\Theta(t)$ satisfying

$$\alpha_i(t) - R(t) = \sum_{j=1}^{d}\sigma_{i,j}(t)\Theta_j(t)$$

From Girsanov Thm, construct the corresponding risk-neutral measure $\tilde{\mathbb{P}}$.

$$\tilde{W}_j(t) = W_j(t) + \int_0^t \Theta_j(u)\mathrm{d}u$$

Second Fundamental Thm of Asset Pricing Assume that market price of risk equations are solvable (*i.e.,* there exists a risk-neutral measure). Then multidimensional market model is complete *i.e.,* Every derivative security can be hedged by trading in the primary assets and the money market account.

Martingale property/Risk neutrality Discounted portfolio value is martingale under $\tilde{\mathbb{P}}$ as are the discounted asset prices. It is noteworthy to remember

$$\mathrm{d}\left(D(t)X(t)\right) = \sum_{i=1}^{m}\Delta_i(t)\mathrm{d}\left(D(t)S_i(t)\right)$$

Risk-neutral measure are constructed to enforce primary assets have their discounted prices to be martingale. Derivative assets will inherit the same property through risk-neutral pricing.

6.2 Numeraire

Money market account numeraire Measure $\tilde{\mathbb{P}}$ is called risk neutral for money market account as $D(t)S(t)$ is a martingale and it is $S(t)$'s price denominated in terms of monery market account

$$D(t)S(t) = \frac{S(t)}{M(t)}$$

Numeraire $\leftrightarrow$ Risk neutral measure When we change numeraire, we need to change the measure in order to maintain the risk neutrality.

$$\begin{aligned} \text{Domestic money market account } &\leftrightarrow \tilde{\mathbb{P}} \\ \text{Foreign money market account } &\leftrightarrow \mathbb{P}^f \\ \text{A zero coupon bound } &\leftrightarrow \mathbb{P}^T \end{aligned}$$

Thm (Stochastic representation of assets) Let $N(t)$ be a non-dividend paying primary or derivative asset. There exists a volatility vector process $\nu(t) = (\nu_1(t), \cdots, \nu_d(t))$ such that

$$\mathrm{d}N(t) = \underbrace{R(t)}_{\text{rate of return}} N(t)\mathrm{d}t + \underbrace{\nu(t)}_{\text{realized risk-neutral rate of return}} N(t)\mathrm{d}\tilde{\mathcal{W}}(t)$$

Equivalently,

$$\mathrm{d}\left(D(t)N(t)\right) = D(t)N(t)\nu(t)\mathrm{d}\tilde{\mathcal{W}}(t)$$

Proof: Due to Martingale Representation Theorem

$$\mathrm{d}\left(D(t)N(t)\right) = \sum_{j=1}^{d} \tilde{\Gamma}(t)\mathrm{d}\tilde{W}_j(t)$$

Since $N(t) > 0$, we can define

$$\nu_j(t) = \frac{\tilde{\Gamma}_j(t)}{D(t)N(t)} \quad \square$$

Change of measure Use volatility vector process $\nu(t)$ in Girsanov Thm:

$$\begin{aligned} \tilde{W}_j^{(N)}(t) &= -\int_0^t \nu_j(u)\mathrm{d}u + \tilde{W}_j(t) \\ \tilde{\mathbb{P}}^{(N)}(A) &= \int_A \frac{D(T)N(T)}{N(0)}\mathrm{d}\tilde{\mathbb{P}}, \quad \forall A \in \mathcal{F} \\ \tilde{\mathbb{E}}^{(N)}X &= \tilde{\mathbb{E}}\left[\frac{XD(T)NT)}{N(0)}\right] \\ \tilde{\mathbb{E}}^{(N)}\left[Y|\mathcal{F}(s)\right] &= \frac{1}{D(s)N(s)}\tilde{\mathbb{E}}\left[YD(t)N(t)|\mathcal{F}(s)\right] \text{ Here } Y \in \mathcal{F}(t) \text{ and } s \le t \end{aligned}$$

Thm (Change of risk-neutral measure) Let $N(t)$ be a non-divided paying asset. Then

$$\begin{aligned}
\mathrm{d}D(t)S(t) &= D(t)S(t)\sigma(t)\mathrm{d}\tilde{W}(t) \\
\mathrm{d}D(t)N(t) &= D(t)N(t)\nu(t)\mathrm{d}\tilde{W}(t) \\
S^{(N)}(t) &= \frac{S(t)}{N(t)} \\
\mathrm{d}S^{(N)}(t) &= S^{(N)}(t)\left[\sigma(t) - \nu(t)\right]\mathrm{d}\tilde{W}^{(N)}(t)
\end{aligned}$$

Thm (Division of two martingales) $M_1(t), M_2(t)$ martingale under $\mathbb{P}$ and $M_2(t) > 0$. $\frac{M_1(t)}{M_2(t)}$ is a martingale under $\mathbb{P}^{(M_2)}$ where

$$\mathbb{P}^{(M_2)}(A) = \int_A \frac{M_2(T)}{M_2(0)}\mathrm{d}\mathbb{P}.$$

6.3 Foreign & domestic risk-neutral measures

Market has two currencies and is driven by two Brownian motions

$$W(t) = (W_1(t), W_2(t))$$

Price of $S(t)$ in domestic currency satisfies

$$\mathrm{d}S(t) = \alpha(t)S(t)\mathrm{d}t + \sigma_1(t)S(t)\mathrm{d}W(t)$$

Moreover,

$$\begin{aligned}
M(t) &= \exp\left(\int_0^t R(u)\mathrm{d}u\right) \text{ Domestic money market account price} \\
D(t) &= \exp\left(-\int_0^t R(u)\mathrm{d}u\right) \text{ Domestic discount process} \\
M^f(t) &= \exp\left(\int_0^t R^f(u)\mathrm{d}u\right) \text{ Foreign money market account price} \\
D^f(t) &= \exp\left(-\int_0^t R^f(u)\mathrm{d}u\right) \text{ Foreign discount process} \\
\mathrm{d}Q(t) &= \gamma(t)Q(t)\mathrm{d}t + \sigma_2(t)Q(t)\left[\rho(t)\mathrm{d}W_1(T) + \sqrt{1-\rho^2(t)}\mathrm{d}W_2(t)\right] \text{ dom. cur. per for. cur.}
\end{aligned}$$

$$\frac{\mathrm{d}S(t)}{S(t)} \cdot \frac{\mathrm{d}Q(t)}{Q(t)} = \rho(t)\sigma_1(t)\sigma_2(t)\mathrm{d}t \;\; \rho(t)\text{: instan. corr bet. relative changes in } S(t), Q(t)$$

6.4 Domestic risk-neutral measure

The following **three** assets can be traded:

- Domestic money market account
- Stock

- Foreign money market account

For each of these assets, to obtain their prices in units of domestic money market account, we will

Step 1: Price it

Step 2: Discount at the domestic interest rate

The resulting value will be a martingale for each. We have

Domestic money market account → Price = 1 which is a martingale under any measure

Stock

$$\begin{aligned} \mathrm{d}S(t) &= \alpha(t)S(t)\mathrm{d}t + \sigma_1(t)S(t)\mathrm{d}W_1(t) \\ &= \sigma_1(t)D(t)S(t)\mathrm{d}\tilde{W}_1(t) \end{aligned}$$

Here $\mathrm{d}\tilde{W}_1(t) = \Theta_1(t) + \mathrm{d}W_1(t)$

$$\sigma_1(t)\Theta_1(t) = \alpha(t) - R(t) \textbf{ 1st market price of risk eq.}$$

Third asset Invest in foreign money market account and convert that into domestic currency and then discount it at the domestic interest rate.

$$\begin{aligned} \mathrm{d}D(t)M^f(t)Q(t) &= D(t)M^f(t)Q(t) \cdot \left[\left(R^f(t) - R(t) + \gamma(t)\right)\mathrm{d}t + \sigma_2(t)\rho(t)\mathrm{d}W_1(t) + \sigma_2(t)\sqrt{1-\rho(t)^2}\mathrm{d}W_2(t)\right] \\ &= D(t)M^f(t)Q(t) \cdot \left[\sigma_2(t)\rho(t)\mathrm{d}\tilde{W}_1(t) + \sigma_2(t)\sqrt{1-\rho(t)^2}\mathrm{d}\tilde{W}_2(t)\right] \\ &= \sigma_2(t)D(t)M^f(t)Q(t)\mathrm{d}\tilde{W}_3(t) \end{aligned}$$

Here we need to have

$$\sigma_2(t)\rho(t)\Theta_1(t) + \sigma_2(t)\sqrt{1-\rho(t)^2}\Theta_2(t) = R^f(t) - R(t) + \gamma(t) \textbf{ 2nd market price of risk eq.}$$

We need first and second market price of risk eqs to have a unique solution to guarantee the existence of a unique risk-neutral measure. Under this measure, the following three processes are martingale:

$$1, D(t)S(t), D(t)M^f(t)Q(t)$$

Exchange rate is an asset? Under domestic risk-neutral measure

$$\mathrm{d}Q(t) = Q(t) \cdot \left[\left(R(t) - R^f(t)\right)\mathrm{d}t + \sigma_2(t)\mathrm{d}\tilde{W}_3(t)\right]$$

$Q(t)$ is a dividend-paying asset. The unit of currency must be invested into the foreign money market account which pays-out a continues dividend at rate $R^f(t)$. If this dividend is reinvested then the process will be $M^f(t)Q(t)$ which has mean rate of return $R(t)$ under the said measure.

6.5 FACT TWO

The following snippet is from Brigo and Mercurio's book:

FACT TWO. *The time-t risk neutral price*

$$Price_t = E_t^{\boxed{B}}\left[\boxed{B(t)}\frac{Payoff(T)}{\boxed{B(T)}}\right]$$

is invariant by change of numeraire: If S is any other numeraire, we have

$$Price_t = E_t^{\boxed{S}}\left[\boxed{S_t}\frac{Payoff(T)}{\boxed{S_T}}\right].$$

In other terms, if we substitute the three occurrences inside the boxes of the original numeraire with a new numeraire the price does not change. This

6.6 Option pricing with a random interest rate

Fixed income derivatives Options on bonds, interest-rate-dependent instruments

Interest rate We assume that $R(t)$ is random

Volatility σ is the fixed volatility for forward prices

$$\mathrm{d}\operatorname*{For}_S(t,T) = \sigma \operatorname*{For}_S(t,T)\mathrm{d}\tilde{W}^T(t)$$

Constant $R(t)$ If $R(t) \equiv r$, then $\mathrm{For}_S(t,T) = e^{r(T-t)}S(t)$. Then

$$D(t)B(t,T) = D(T)B(T,T).$$

Thus, $\sigma^*(t,T) = 0$ and $\tilde{W}^T(t) = \tilde{W}(t)$. Therefore,

$$\begin{aligned}\mathrm{d}\operatorname*{For}_S(t,T) = e^{rT}\mathrm{d}D(t)S(t) = \\ = \sigma e^{rT}D(t)S(t)\mathrm{d}\tilde{W}(t)\end{aligned}$$

i.e, $S(t)$ has constant volatility under $\tilde{W}$ if interest rate is constant. Therefore, in this case

$$\mathrm{d}D(t)S(t) = \sigma D(t)S(t)\mathrm{d}\tilde{W}(t)$$

BSM for random interest rate Value of a European call at time t is computed as below:

$$\begin{aligned}V(t) &= S(t)N(d_+(t)) - KB(t,T)N(d_-(t)) \\ d_\pm(t) &= \frac{1}{\sigma\sqrt{\tau}}\left[\frac{1}{K}\log\operatorname*{For}_S(t,T) \pm \frac{1}{2}\sigma^2 t\right]\end{aligned}$$

Proof We begin by noting that

$$\begin{aligned}\underset{S}{\text{For}}(t,T) &= \underset{S}{\text{For}}(0,T)\exp\left(\sigma\tilde{W}^T(t) - \frac{1}{2}\sigma^2 t\right)\\ &= \frac{S(0)}{B(0,T)}\exp\left(\sigma\tilde{W}^T(t) - \frac{1}{2}\sigma^2 t\right)\end{aligned}$$

Taking $S(t)$ to be the numeraire,

$$\tilde{\mathbb{P}}^S(A) = \frac{1}{S(0)}\int_A D(T)S(T)\mathrm{d}\tilde{\mathbb{P}}, \quad \forall A \in \mathcal{F}.$$

Then

$$\begin{aligned}\frac{B(t,T)}{S(t)} &= \frac{1}{\text{For}_S(t,T)}\\ \mathrm{d}\left(\frac{1}{\text{For}_S(t,T)}\right) &= -\frac{\sigma}{\text{For}_S(t,T)}\left(-\sigma\mathrm{d}t + \mathrm{d}\tilde{W}^T(t)\right)\\ &= -\frac{\sigma}{\text{For}_S(t,T)}\mathrm{d}\tilde{W}^S(t)\end{aligned}$$

$\frac{B(t,T)}{S(t)}$ is a martingale under $\tilde{\mathbb{P}}^S$ and thus $\tilde{W}^S$ is a Brownian motion under $\tilde{\mathbb{P}}^S$ ($\mathrm{d}\tilde{W}^S\mathrm{d}\tilde{W}^S = \mathrm{d}t$).
Next

$$\frac{1}{\text{For}_S(t,T)} = \frac{B(0,T)}{S(0)}\exp\left(-\sigma\tilde{W}^S(t) - \frac{1}{2}\sigma^2 t\right)$$

Risk-neutral pricing gives

$$\begin{aligned}V(0) &= \tilde{\mathbb{E}}[D(T)\left(S(T)-K\right)^+]\\ &= S(0)\tilde{\mathbb{E}}\left[\frac{D(T)S(T)}{S(0)}\mathbf{1}_{\{S(T)>K\}}\right] - KB(0,T)\tilde{\mathbb{E}}\left[\frac{D(T)}{B(0,T)}\mathbf{1}_{\{S(T)>K\}}\right]\\ &= S(0)\tilde{\mathbb{E}}^S\left[\mathbf{1}_{\{S(T)>K\}}\right] - KB(0,T)\tilde{\mathbb{E}}^T\left[\mathbf{1}_{\{S(T)>K\}}\right]\\ &= S(0)\tilde{\mathbb{P}}^S\left(S(T)>K\right) - KB(0,T)\tilde{\mathbb{P}}^T\left(S(T)>K\right)\\ &= S(0)\tilde{\mathbb{P}}^S\left(\frac{1}{\text{For}_S(T,T)} < \frac{1}{K}\right) - KB(0,T)\tilde{\mathbb{P}}^T\left(\underset{S}{\text{For}}(T,T) > K\right)\\ &= S(0)N(d_+(0)) - KB(0,T)N(d_-(0)).\end{aligned}$$

Denomination in zero-coupon bonds gives

$$\frac{V(t)}{B(t,T)} = \underset{S}{\text{For}}(t,T)N(d_+(t)) - KN(d_-(t)).$$

Hedge a short position To hedge, do

- Hold $N(d_+(t))$ of the asset at time t
- Short $KN(d_-(t))$ zero coupon bond at time t

Self-financing The associated capital gains differentials is $N(d_+(t))\mathrm{d}\,\mathrm{For}_S(t,T)$. We have

$$\mathrm{d}\left(\frac{V(t)}{B(t,T)}\right) = N(d_+(t))\mathrm{d}\,\underset{S}{\mathrm{For}}(t,T) + \underset{S}{\mathrm{For}}(t,T)\mathrm{d}N(d_+(t)) + \mathrm{d}\,\underset{S}{\mathrm{For}}(t,T)\mathrm{d}N(d_+(t)) - K\mathrm{d}N(d_-(t)).$$

For self-financing to hold, we need to ensure that

$$\underset{S}{\mathrm{For}}(t,T)\mathrm{d}N(d_+(t)) + \mathrm{d}\,\underset{S}{\mathrm{For}}(t,T)\mathrm{d}N(d_+(t)) = K\mathrm{d}N(d_-(t))$$

This holds and thus there is no need to infuse cash to maintain the position.

7 Term-Structure Models

Boostrap for yield curve Imply yields to different maturities using market's bonds prices.

$$\text{price of zero-coupon bond} = \text{face value} \times e^{-\text{yield}\times\text{time to maturity}} \Rightarrow \text{a maturity-yield pair}$$

Short rate Interest rate ≡ Short rate. The following is taken from Brigo and Mercurio book. Here AB(O) stands for Analytical Bond (Option) price.

Model	Dynamics	$r>0$	$r\sim$	AB	AO
V	$dr_t = k[\theta - r_t]dt + \sigma dW_t$	N	$\mathcal{N}$	Y	Y
CIR	$dr_t = k[\theta - r_t]dt + \sigma\sqrt{r_t}dW_t$	Y	NCχ^2	Y	Y
D	$dr_t = ar_t dt + \sigma r_t dW_t$	Y	L$\mathcal{N}$	Y	N
EV	$dr_t = r_t\left[\eta - a\ln r_t\right]dt + \sigma r_t dW_t$	Y	L$\mathcal{N}$	N	N
HW	$dr_t = k[\theta_t - r_t]dt + \sigma dW_t$	N	$\mathcal{N}$	Y	Y
BK	$dr_t = r_t\left[\eta_t - a\ln r_t\right]dt + \sigma r_t dW_t$	Y	L$\mathcal{N}$	N	N
MM	$dr_t = r_t\left[\eta_t - \left(\lambda - \frac{\gamma}{1+\gamma t}\right)\ln r_t\right]dt + \sigma r_t dW_t$	Y	L$\mathcal{N}$	N	N
CIR++	$r_t = x_t + \varphi_t,\ \ dx_t = k[\theta - x_t]dt + \sigma\sqrt{x_t}dW_t$	Y*	SNCχ^2	Y	Y
EEV	$r_t = x_t + \varphi_t,\ dx_t = x_t[\eta - a\ln x_t]dt + \sigma x_t dW_t$	Y*	SL$\mathcal{N}$	N	N

Table 3.1. Summary of instantaneous short rate models.

Multi-factor models PDEs satisfied by zero-coupon bonds in one-factor short-rate (*e.g.*, HW, CIR) models could be extended to multi-factors.

Abstract factors In multi-factor models we start with abstract factors. Recall interest rate is not an asset and market price of risk cannot be obtained.

Calibration Multi-factor models are calibrated to market prices for zero-coupon bonds or some fixed income derivatives.

HJM States are $f(t,T)$ *i.e.*, instantaneous rate to lock at time t to borrow at time T

Forward rate curve $T \to f(t,T)$ is called the forward rate curve

BSM(market model) States are $L(t,T)$ *i.e.,* simple rate to lock at time t to borrow between T and $T+\delta$. Often $\delta = 0.25$ or 3 month LIBOR.

$$1 + \delta L(t,T) = \frac{B(t,T)}{B(t,T+\delta)}$$

Affine Yields

$$B(t,T) = e^{-R(t)C(t,T)-A(t,T)}$$

Two-factor Vasicek Model

$$\begin{aligned}
\mathrm{d}X_1(t) &= (a_1 - b_{11}X_1(t) - b_{12}X_2(t))\,\mathrm{d}t + \sigma_1 \mathrm{d}\tilde{B}_1(t) \\
\mathrm{d}X_2(t) &= (a_2 - b_{21}X_1(t) - b_{22}X_2(t))\,\mathrm{d}t + \sigma_2 \mathrm{d}\tilde{B}_2(t) \\
R(t) &= \epsilon_0 + \epsilon_1 X_1(t) + \epsilon_2 X_2(t) \\
B &\succeq 0 \iff X_1, X_2 \text{mean-reverting}
\end{aligned}$$

Canonical two-factor Vasicek model

$$\begin{aligned}
\mathrm{d}Y_1(t) &= -\lambda_1 Y_1(t)\mathrm{d}t + \mathrm{d}\tilde{W}_1(t) \\
\mathrm{d}Y_2(t) &= -\lambda_{21} Y_1(t)\mathrm{d}t - \lambda_2 Y_2(t)\mathrm{d}t + \mathrm{d}\tilde{W}_2(t) \\
R(t) &= \delta_0 + \delta_1 Y_1(t) + \delta_2 Y_2(t) \\
\lambda_1&, \lambda_2 > 0
\end{aligned}$$

Long rate $L(t)$ Yield of zero-coupon bond maturing at $\bar{\tau} + t$

Gaussian factor processes

$$\begin{aligned}
\mathrm{d}\boldsymbol{Y}(t) &= -\Lambda \boldsymbol{Y}(t) + \mathrm{d}\tilde{W}(t) \\
\boldsymbol{Y}(t) &= e^{-\Lambda t}\boldsymbol{Y}(0) + \int_0^t e^{-\Lambda(t-u)}\mathrm{d}\tilde{W}(u) \\
Y_1(t), Y_2(t), R(t) &\text{ are normally distributed!}
\end{aligned}$$

Two-factor CIR model

$$\begin{aligned}
\mathrm{d}Y_1(t) &= (\mu_1 - \lambda_{11}Y_1(t) - \lambda_{12}Y_2(t))\,\mathrm{d}t + \sqrt{Y_1(t)}\mathrm{d}\tilde{W}_1(t) \\
\mathrm{d}Y_2(t) &= (\mu_2 - \lambda_{21}Y_1(t) - \lambda_{22}Y_2(t))\,\mathrm{d}t + \sqrt{Y_2(t)}\mathrm{d}\tilde{W}_2(t) \\
R(t) &= \delta_0 + \delta_1 Y_1(t) + \delta_2 Y_2(t) \\
\mu_i \geq 0, &\quad \lambda_{ii} > 0, \lambda_{ij} < 0, \delta_0 \geq 0, \delta_1, \delta_2 > 0
\end{aligned}$$

Canonical two-factor mixed model

$$\begin{aligned}
\mathrm{d}Y_1(t) &= (\mu - \lambda_1 Y_1(t))\,\mathrm{d}t + \sqrt{Y_1(t)}\mathrm{d}\tilde{W}_1(t) \\
\mathrm{d}Y_2(t) &= -\lambda_2 Y_2(t)\mathrm{d}t + \sigma_{21}\sqrt{Y_1(t)}\mathrm{d}\tilde{W}_1(t) + \sqrt{\alpha + \beta Y_1(t)}\mathrm{d}\tilde{W}_2(t) \\
R(t) &= \delta_0 + \delta_1 Y_1(t) + \delta_2 Y_2(t) \\
\mu, \alpha&, \beta \geq 0, \lambda_i > 0
\end{aligned}$$

Bond prices

$$B(t,T) = f(t, Y_1(t), Y_2(t))$$
$$\mathrm{d}D(t)B(t,T) = [\underbrace{\cdots}_{\text{set}=0}]\mathrm{d}t + [\cdots]\mathrm{d}\tilde{W}_1(t) + [\cdots]\mathrm{d}\tilde{W}_2(t)$$

Forward rates At time t

- **Short** 1 $T-$maturity bonds. Receive $B(t,T)$
- **Long** $\frac{B(t,T)}{B(t,T+\delta)}$, $T+\delta$-maturity bonds. Pay $B(t,T)$

Later,

- At T, pay 1.
- At $T+\delta$. Receive $\frac{B(t,T)}{B(t,T+\delta)}$

The yield that explains this surplus is equal to $\frac{1}{\delta}\log\frac{B(t,T)}{B(t,T+\delta)}$. Define

$$f(t,T) = \lim_{\delta\to 0}\frac{1}{\delta}\log\frac{B(t,T)}{B(t,T+\delta)} = -\frac{\partial}{\partial T}B\log(t,T)$$

Note

$$B(t,T) = \exp\left(-\int_t^T f(t,x)\mathrm{d}x\right), \quad 0 \le t \le T \le \bar{T}.$$

HJM Assume the *initial forward rate curve* $f(0,T)$ for $0 \le T \le \bar{T}$ is given at time 0.

$$\begin{aligned}
\mathrm{d}f(t,T) &= \alpha(t,T)\mathrm{d}t + \sigma(t,T)\mathrm{d}W(t) \\
\mathrm{d}B(t,T) &= B(t,T)\left[R(t) - \alpha^*(t,T) + \tfrac{1}{2}\sigma^*(t,T)^2\right]\mathrm{d}t - \sigma^*(t,T)B(t,T)\mathrm{d}W(t) \\
\alpha^*(t,T), \sigma^*(t,T) &= \int_t^T \alpha(t,x)\mathrm{d}x, \int_t^T \sigma(t,x)\mathrm{d}x \\
\mathrm{d}D(t)B(t,T) &= D(t)B(t,T)\left[-\sigma^*(t,T)\left[\Theta(t)\mathrm{d}t + \mathrm{d}W(t)\right]\right] \\
-\alpha^*(t,T) + \tfrac{1}{2}\sigma^*(t,T)^2 &= -\sigma^*(t,T)\Theta(t) \\
\tilde{W}(t) &= \int_0^t \Theta(u)\mathrm{d}u + W(t) \\
D(t)B(t,T) &= -D(t)B(t,T)\sigma^*(t,T)\mathrm{d}\tilde{W}(t)
\end{aligned}$$

As long as $\sigma(t,T) \neq 0$, Θ is unique and hence HJM is complete. Therefore, all interest rate derivatives can be hedged by trading in zero-coupon bonds.

Term-Structure evolution under risk-neutral measure

$$\mathrm{d}f(t,T) = \sigma(t,T)\sigma^*(t,T)\mathrm{d}t + \sigma(t,T)\mathrm{d}\tilde{W}(t)$$
$$\mathrm{d}B(t,T) = R(t)B(t,T)\mathrm{d}t - \sigma^*(t,T)B(t,T)\mathrm{d}\tilde{W}(t)$$

Affine is HJM Will follow immediately from the relationship between $f(t,T)$ and $B(t,T)$.

No arbitrage condition for Affine Assume that

$$\begin{aligned} \mathrm{d}R(t) &= \beta(t,R(t))\mathrm{d}t + \gamma(t,R(t))\mathrm{d}\tilde{W}(t) \\ B(t,T) &= e^{-R(t)C(t,T)-A(t,T)} \end{aligned}$$

The no arbitrage condition is follows.

$$\frac{\partial}{\partial T}C(t,T)\beta(t,R(t)) + R(t)\frac{\partial}{\partial T}C(t,T) + \frac{\partial}{\partial T}A'(t,T) = \left(\frac{\partial}{\partial T}C(t,T)\right)C(t,T)\gamma(t,R(t))^2$$

HJM and log-normal returns In order to adapt BSM formula for equity options for use in fixed income markets, we wish to have $f(t,T)$ log-normal under risk-neutral measure *i.e.,* $\sigma(t,T) = \sigma f(t,T)$. For T near t,

$$\sigma^*(t,T) = \int_t^T \sigma(t,x)\mathrm{d}x = \sigma\int_t^T f(t,x)\mathrm{d}x \approx \sigma f(t,T)$$

Thus, the dt-term becomes

$$f'(t) = \sigma^2 f(t,T)^2 \Rightarrow f'(t) = \frac{\sigma^2 f^2(0)}{(1-\sigma^2 f(0)t)^2}$$

7.1 T-forward measure

Define the T-forward measure as below:

$$\tilde{\mathbb{P}}^T(A) = \frac{1}{B(0,T)}\int_A D(T)\mathrm{d}\tilde{\mathbb{P}} \text{ for all } A \in \mathcal{F}$$

Note that

$$\mathrm{d}B(t,T) = R(t)B(t,T)\mathrm{d}t - \sigma^*(t,T)B(t,T)\mathrm{d}\tilde{W}(t)$$

The following is a Brownian motion under $\tilde{\mathbb{P}}^T$

$$\tilde{W}^T(t) = \int_0^t \sigma^*(u,T)\mathrm{d}u + \tilde{W}(t)$$

All assets denominated in zero-coupon bonds maturing at time T are martingale under $\tilde{\mathbb{P}}^T$. Finally,

$$V(t) = B(t,T)\tilde{\mathbb{E}}^T\left[V(T)|\mathcal{F}(t)\right]$$

Theorem The following identity holds

$$f(t,T) = \tilde{\mathbb{E}}^T\left[R(T)|\mathcal{F}(t)\right]$$

Proof: Recall that

$$B(t,T) = \tilde{\mathbb{E}}\left[e^{-\int_t^T R(s)\mathrm{d}s} | \mathcal{F}(t)\right]$$

Thus

$$\begin{aligned}
\tilde{\mathbb{E}}^T\left[R(T)|\mathcal{F}(t)\right] &= \frac{1}{B(t,T)}\tilde{\mathbb{E}}\left[R(T)e^{-\int_t^T R(s)\mathrm{d}s} | \mathcal{F}(t)\right] \\
&= -\frac{1}{B(t,T)}\tilde{\mathbb{E}}\left[\frac{\partial}{\partial T}e^{-\int_t^T R(s)\mathrm{d}s} | \mathcal{F}(t)\right] \\
&= -\frac{1}{B(t,T)}\frac{\partial}{\partial T}\tilde{\mathbb{E}}\left[e^{-\int_t^T R(s)\mathrm{d}s} | \mathcal{F}(t)\right] \\
&= -\frac{1}{B(t,T)}\frac{\partial}{\partial T}B(t,T) \\
&= f(t,T).
\end{aligned}$$

8 Default-free setting (ref: notes by Grasselli and Hurd)

Default-free zero-coupon bonds Denote by $P_t(T)$ the price of a default-free zero paying 1 at maturity at time $t \leq T$.

Assumption on market condition

- Bond market is frictionless
  - No transaction cost
  - Zero bid-ask spread
  - Small trades do not move the market
  - Unlimited short selling
- Arbitrage-free bonds for all maturity $T > t$ exists
- $P_t(T_1) \geq P_t(T_2)$ for $T_1 \leq T_2$

Proposition Let $t < S < T$. A payment of $P_S(T)$ at time S is valued $P_t(T)$ at time t.

Proposition Let X be $\mathcal{F}_t$-measurable. A payment of X at time T is valued $XP_t(T)$ at time t.

Different notions of interest rate in terms of zero-coupon bonds

- Forward rate

$$P_t(S) = P_t(T)\exp\left(\underbrace{R_t(S,T)}_{\text{cont. compounded}}(T-S)\right) = P_t(T)\left[1 + \underbrace{L_t(S,T)}_{\text{simply compounded}}(T-S)\right]$$

- Yield (forward rates when $S = t$)

$$1 = \exp\left(\underbrace{R_t(T)}_{\text{cont. compounded}}(T-t)\right)P_t(T) = \left[1 + \underbrace{L_t(T)}_{\text{simply compounded}}(T-t)\right]P_t(T)$$

LIBOR A prime example of simply compounded rates

Lemma A payment of $L_S(T)$ at time T is valued $L_t(S,T)P_t(T)$ at time t

Proof $L_S(T) \in \mathcal{F}(S)$ and so $L_S(T)P_S(T)$ is the payment's value at time S. But

$$\begin{aligned} L_S(T)P_S(T) &= \frac{1}{T-S} - \frac{P_S(T)}{T-S} \\ &= \frac{P_S(S)}{T-S} - \frac{P_S(T)}{T-S} \end{aligned}$$

Use propositions above, value at time t of this payment therefore is

$$\frac{P_t(S)}{T-S} - \frac{P_t(T)}{T-S} = L_t(S,T)P_t(T)$$

Instantaneous rates

- Forward rates

$$f_t(T) = \lim_{S \to T-} L_t(S,T) = \lim_{S \to T-} R_t(S,T) = -\frac{\partial \log P_t(T)}{\partial T}$$

- Spot rates

$$r_t = \lim_{T \to t+} L_t(T) = \lim_{T \to t+} R_t(T) \text{ (note: } f_t(t) = r_t)$$

Money-market account & Stochastic discount factor Money-market account is a tradable asset and satisfies

$$\mathrm{d}C_t = r_t C_t \mathrm{d}t$$

Stochastic discount factor is defined as below

$$D(t,T) = \frac{C_t}{C_T} = \exp\left(-\int_t^T r_s \mathrm{d}s\right)$$

Risk-neutral pricing Price processes $\{Y_t^i\}_{i \in \mathcal{I}}$ for non-dividend paying assets will be arbitrage-free if there exists some risk-neutral measure Q such that $C_t^{-1} Y_t^i$ is a Q-martingale. In particular, if $Y_t = P_t(T)$, then

$$P_t(T) = \mathbb{E}^Q \left[D(t,T) | \mathcal{F}_t\right]$$

Bootstrapped interest rate model Recall that $-\log P_t(T) = \int_t^T f_t(s)\mathrm{d}s$. Suppose zero-coupon bonds of maturities $T_1, \cdots, T_N$ are traded with $\Delta_n = T_n - T_{n-1}$ at time 0. We have

$$\begin{aligned}
-\log P_0(T_1) &= f_1 \Delta_1 \\
-\log P_0(T_2) &= f_1 \Delta_1 + f_2 \Delta_2 \\
&\vdots \\
-\log P_0(T_N) &= f_1 \Delta_1 + \cdots + f_N \Delta_N
\end{aligned}$$

Solve for f_i and for $t \leq T_N$, define the bootstrapped forward curve found at time 0

$$f_0(t) = f_k \text{ where } T_{k-1} \leq t \leq T_k$$

Coupon-paying bonds c_k (deterministic) is paid at T_k for $k = 1, \cdots, N$. Then

$$\text{Coupon's worth at time } t \text{ is } = \sum_{k=1}^{n} c_k P_t(T_k)$$

Floating-rate notes In this case, c_k is not deterministic. A common example is where

$$c_k = L_{k-1}(T_k)(T_k - T_{k-1})\mathcal{N} \text{ for } k = 1, \cdots, N-1$$

And $c_N = L_{N-1}(T_N)(T_N - T_{N-1})\mathcal{N} + \mathcal{N}$. Here we also assume that $T_0 = 0$ or right after a contractual payment. A payment of

$$L_{k-1}(T_k)(T_k - T_{k-1}) = \frac{1}{P_{T_{k-1}}(T_k)} - 1 \in \mathcal{F}_{T_{k-1}}$$

at T_k is worth $1 - P_{T_{k-1}}(T_k)$ at T_{k-1} and finally $P_t(T_{k-1}) - P_t(T_k)$ at t. Thus, the value of this floating-rate note at time 0 is $\mathcal{N}$.

Forward rate agreement In a forward rate agreement, we have

- Period $[S, T]$ for $S > t$
- Notional $\mathcal{N}$
- Agreed upon simple interest rate K

The borrower receives $\mathcal{N}$ at S and repays $\mathcal{N}(1 + K(T - S))$ at time T. Therefore,

- Receives $\mathcal{N}(1 + L_S(T)(T - S))$
- Pays $\mathcal{N}(1 + K(T - S))$

The value of this cash flow at time t is

$$\mathcal{N}\underbrace{[P_t(S) - P_t(T)(1 + K(T - S))]}_{\in \mathcal{F}_t} = \mathcal{N}P_t(T)[L_t(S, T) - K](T - S)$$

Interest rate swap Consider the dates $\mathcal{N} = (T_1, \cdots, T_N)$ and let the cash flow at T_k be

$$\mathcal{N}L_{T_{k-1}}(T_k)(T_k - T_{k-1}) - \mathcal{N}K(T_k - T_{k-1})$$

Value at 0 of this payment is

$$\mathcal{N}(P_0(T_{k-1}) - P_0(T_k)) - \mathcal{N}K(T_k - T_{k-1})P_0(T_k)$$

Summing up, we obtain that

$$IRS(\mathcal{N}, \mathcal{T}, K) = \mathcal{N}\left[1 - P_0(T_N) - K\sum_{k=1}^{N} P_0(T_k)(T_k - T_{k-1})\right]$$

8.1 Defaultable setting

Solvent or bankrupt Denote by τ the time of default. Then $\tau > t$ means the company is solvent at time t; otherwise it is bankrupted.

Defaultable zero-coupon bond Denote by $\bar{P}_t(T)$ the time t value of a defautable zero-coupon bond with face value \$1 issued by a specific company with maturity T. Provided that $\mathbb{P}[\tau \leq T | \tau > t] > 0$. it holds that

$$\bar{P}_t(T)\mathbf{1}_{\{\tau > t\}} < P_t(T).$$

Default risky forward rates Assuming that the firm's bonds exists for all maturities $T > t$ and $\bar{P}_t(T)$ is differentiable in T, then define $\bar{f}_t(T)$ by

$$\bar{P}_t(T) = e^{-\int_t^T \bar{f}_t(u)\mathrm{d}u}$$

Credit spread We assume $\bar{f}_t(T) \geq f_t(T)$ almost surely. In other words, the prices of defaultable bonds show a sharper decrease as a function of maturity than do prices of default-free bonds. Thus

$$\underbrace{YS_t}_{\text{Yield spread}}(T) = \frac{1}{T-t}\int_t^T \underbrace{FS_t}_{\text{Forward spread}}(s)\mathrm{d}s = \frac{1}{T-t}\int_t^T \left(\bar{f}_t(s) - f_t(s)\right)\mathrm{d}s = \frac{1}{T-t}\log\frac{P_t(T)}{\bar{P}_t(T)}$$

Defaultable LIBOR rate Simply compounded defaultable forward rate or defaultable LIBOR rate is defined by

$$\left[1 + \bar{L}_t(S,T)(T-S)\right]\bar{P}_t(T) = \bar{P}_t(S).$$

Defaultable floating-rate note (*e.g.*, par floater) Under the assumption of zero recovery on the bond at default, the payment stream is

$$\begin{aligned} c_k &= \left[L_{T_{k-1}}(T_k) + s^{PF}\right](T_k - T_{k-1})\mathbf{1}_{\{\tau > T_k\}}\mathcal{N}, \quad k = 1, \cdots, N-1 \\ c_N &= \left[L_{T_{N-1}}(T_N) + s^{PF}\right](T_N - T_{N-1})\mathbf{1}_{\{\tau > T_N\}}\mathcal{N} \end{aligned}$$

9 XVA

Consider a security that pays $V(T)$ at time T. The t-price (t-value) of a security that pays $V(T)$ at time T is equal to

$$V(t) = \frac{1}{D(t)}\tilde{\mathbb{E}}\left[D(T)V(T)|\mathcal{F}_t\right] \qquad \text{(Risk-neutral Pricing)}$$

$\tilde{\mathbb{E}}$ is the unique risk-neutral measure (by assumption) and $\mathcal{F}_t$ is the Brownian motion generated filtration. Moreover, $D(t)$ is the discount process

$$D(t) := e^{-\int_0^t R(s)\mathrm{d}s}$$

Under Counterparty Credit Risk (CCR), derivative pricing takes an important twist as will be discussed below.

Default time: Throughout, we denote by τ_C the time when counterparty C defaults. τ_C is a stopping time w.r.t. $\{\mathcal{F}_t\}_{t\geq 0}$. Distribution of τ_C will be discussed in Section **??**.

Credit Exposure Loss in the event of counterparty's default at time t is called *exposure* at time t and is calculated as below:

$$E(t) = \max(V(t), 0) \qquad \text{(Positive Exposure)}$$

Negative exposure is similarly calculated by replacing $V(t)$ with $-V(t)$ in the last displayed equation. Negative exposure at time t is the amount, we owe to the counterparty C, if it defaults at time t. *Expected (positive) exposure* at t is defined as below:

$$EE(t) := \tilde{\mathbb{E}}\left[E(t)|\mathcal{F}_t\right]$$

Given a confidence level p, potential future exposure at time t is defined as

$$PFE(t) := \inf\{x : \tilde{\mathbb{P}}(E(t) \geq x) \leq p\}$$

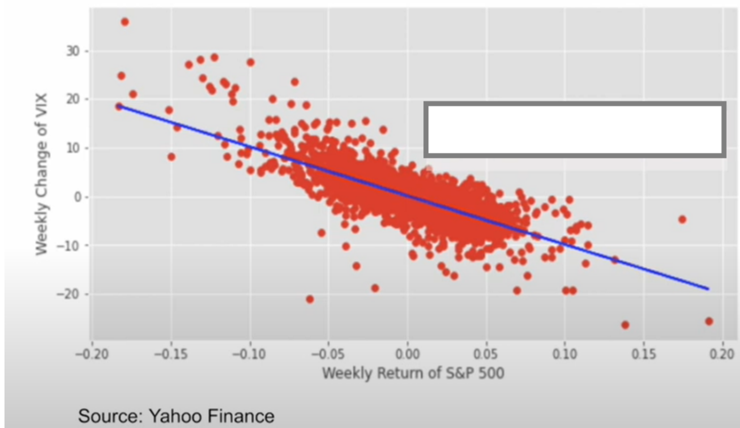
10 SABR Model

In Black-Scholes model implied volatility is assumed to be constant. Namely,

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t$$

VIX Market's expectation for 1 year S&P 500 - directionless. For example, VIX = 20 means that market expects S&P 500 moves by ±20% over a year. To compute forward 1 month expectation, we consider $\pm\frac{VIX}{\sqrt{12}}$.

Equity & volatility correlation Equity and volatility are negatively correlated in general.



This could be due to

- **Leverage effect**: When stock goes down the leverage of the company increases and hence the equity is more volatile.
- **Risk Aversion:** Persistence high volatility causes the stock prices to drop and enforces the asset managers to sell risky assets.

Stochastic Alpha Beta Rho (SABR) model was proposed by Hagan et al and it models the dynamic of forward prices

$$\begin{aligned} \mathrm{d}F_t &= \alpha_t F_t^{\beta} \mathrm{d}W_t^1 \\ \mathrm{d}\alpha_t &= \nu \alpha_t \mathrm{d}W_t^2 \\ \mathrm{d}W_t^1 \mathrm{d}W_t^2 &= \rho \mathrm{d}t \end{aligned}$$

Except when $\beta \in \{0, 1\}$, no close form solution is known for option pricing under SABR. However, an asymptotic estimation exists in the case where $T\nu^2$ is small.

$$C = e^{-rT}\left(FN(d_1) - KN(d_2)\right)$$

Where

$$F = e^{rT} S$$

$$d_1 = \frac{\ln \frac{F}{K} + \frac{\sigma_B^2 T}{2}}{\sigma_B \sqrt{T}}$$

$$d_2 = d_1 - \sigma_B \sqrt{T}$$

$$\sigma_B(K, F) = \text{a closed form formula}$$

SABR reduces to BSM When $\beta = 1$ and $\nu = 0$, then SABR simplifies to the Black-Scholes model with flat volatility smile. In practice, β is fixed to improve the stability of calibration.

Volatility level Volatility level is controlled using α_0

Kurtosis ν controls the kurtosis of the volatility curve. Smaller ν corresponds to more flat curve.

Volatility skew ρ, spot-vol parameter controls volatility skew. Figure below is taken from the book *Mathematical modeling and computation in finance: with exercises and Python and MATLAB computer codes*

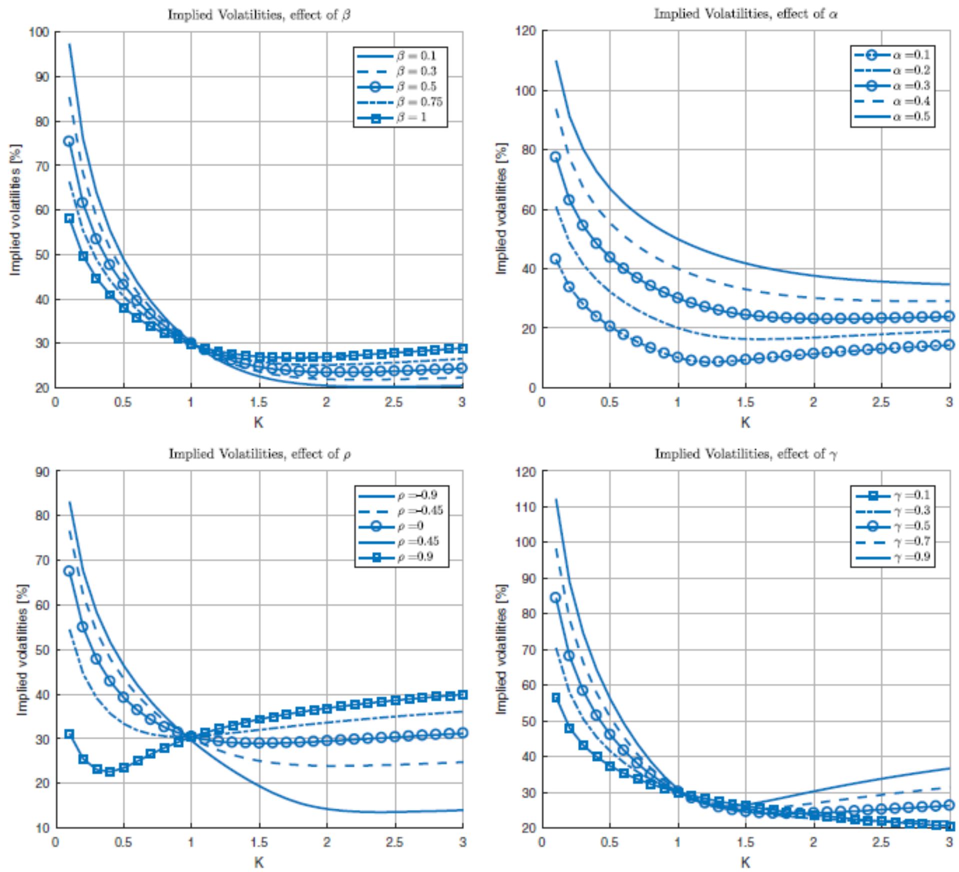


Figure 4.8: *Different implied volatility shapes under Hagan's implied volatility parametrization, depending on different model parameters.*

11 Heston Model

Under real probability P,

$$\begin{aligned}
\mathrm{d}S_t &= \mu S_t \mathrm{d}t + \sqrt{\nu_t} S_t \mathrm{d}W_t^{1,P} \\
\mathrm{d}\nu_t &= \kappa(\theta - \nu_t) + \xi\sqrt{\nu_t}\mathrm{d}W_2^{2,P} \\
\mathrm{d}W_t^{1,P}\mathrm{d}W_2^{2,P} &= \rho \mathrm{d}t.
\end{aligned}$$

Here

$\kappa =$ Speed of mean-reversion. In practice, it's fixed as other params are enough for calibration
$\theta =$ Long-term mean of variance
$\xi =$ Volatility of variance. Control tail-risk & kurtosis. Larger value results in fatter tails
$\rho =$ Control the skewness

Feller condition Instantaneous variance ν_t is strictly positive when the following holds

$$2\kappa\theta > \xi^2$$

BSM reduction Assuming $\theta = \nu_0$ and $\xi = \rho = 0$, Heston reduces to BSM.

Half-life $\frac{\ln(2)}{K}$ Average time it takes to get halfway back to the mean.

11.1 Risk-neutral measure

Under risk-neutral probability Q

$$\begin{aligned}
\mathrm{d}S_t &= rS_t \mathrm{d}t + \sqrt{\nu_t} S_t \mathrm{d}W_t^{1,Q} \\
\mathrm{d}\nu_t &= \kappa^Q(\theta^Q - \nu_t) + \xi\sqrt{\nu_t}\mathrm{d}W_2^{2,Q} \\
\mathrm{d}W_t^{1,Q}\mathrm{d}W_2^{2,Q} &= \rho \mathrm{d}t \\
\kappa^Q &= \kappa + \lambda. \text{ Here } \lambda = \text{Variance Risk Premium} \\
\theta^Q &= \frac{\kappa\theta}{\kappa + \lambda} \\
\mathrm{d}W_t^{1,Q} &= \mathrm{d}W_t^{1,P} + \frac{\mu - r}{\sqrt{\nu_t}}\mathrm{d}t. \text{ Here } \mu - r = \text{Risk Premium and } \frac{\mu - r}{\sqrt{\nu_t}} = \text{Sharp Ratio} \\
\mathrm{d}W_t^{2,Q} &= \mathrm{d}W_t^{2,P} + \frac{\lambda\nu_t}{\xi\sqrt{\nu_t}}\mathrm{d}t. \text{ Here } \lambda = \text{Variance Risk Premium and } \frac{\lambda\nu_t}{\xi\sqrt{\nu_t}} = \text{Market Price of Vol Risk}
\end{aligned}$$

How to know the variance premium λ? No need to bother as Heston's parameters will be directly calibrated from option prices.

Completeness Market is not complete as risk-neutral measure exists but it is not unique. There are two source of random.

Volatility Surface Calibration is done as follows:

$$\left(\widehat{\nu_0}, \widehat{\kappa^Q}, \widehat{\theta^Q}, \widehat{\xi}, \widehat{\rho}\right) = \underset{\nu_0, \kappa^Q, \theta^Q, \xi, \rho}{\operatorname{argmin}} \sum \left(C_{\text{Heston}}\left(S_0, r, K_i, T_i, \nu_0, \kappa^Q, \theta^Q, \xi, \rho\right) - C(K_i, T_i)\right)^2$$

Denote

$$C_{\text{Heston}}\left(S_0, r, K, T, \nu_0, \kappa^Q, \theta^Q, \xi, \rho\right) = C_{BS}\left(S_0, r, K, T, \sigma_{BS}\right)$$

In other words,

$$\sigma_{BS}(K, T) = C_{BS}^{-1}\left(S_0, r, K, T, C_{\text{Heston}}\left(S_0, r, K, T, \nu_0, \kappa^Q, \theta^Q, \xi, \rho\right)\right)$$

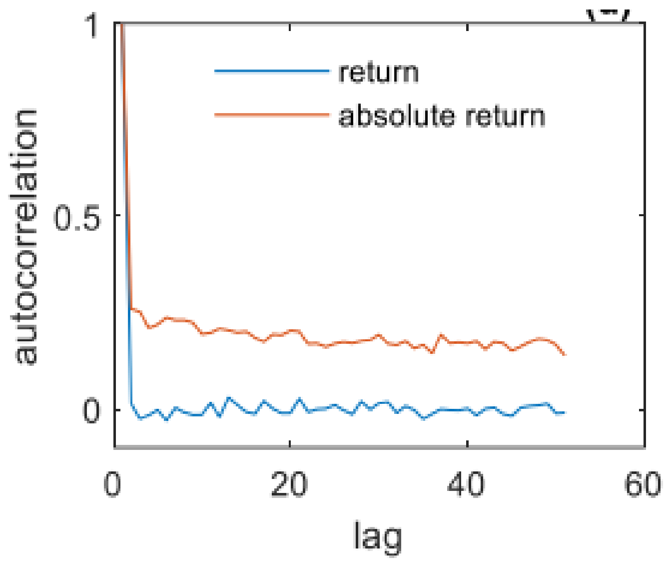


Figure 1: Correlation for stock prices for different lag values

12 Volatility Clustering

Consider an asset price $P(t)$ and denote

$$r(t) = \frac{P(t) - P(t-1)}{P(t-1)}$$

The following two empirical observations hold:

- For some constant C, it holds that

$$\mathbb{P}\left(|r(t)| > x\right) > Cx^{-\alpha}$$

- For considerably different value of h, it holds that

$$\text{Corr}\left(|r(t)|, |r(t+h)|\right) > 0.$$

  Whearas

$$\text{Corr}\left(r(t), r(t+h)\right) \approx 0.$$

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