Exercise 10.1

We know that Itô integrals are Gaussian processes. Since expectations of Itô integrals appearing in the right-hand side of 10.2.44, 10.2.45 and 10.2.46 are all equal to zero, we conclude that

$$\tilde{\mathbb{E}}\left[Y_1(t)\right] = e^{-\lambda_1 \cdot t} \cdot Y_1(0)$$

Additionally, it holds that

$$\tilde{\mathbb{E}}\left[Y_2(t)\right] = \begin{cases} \frac{\lambda_{21}}{\lambda_1 - \lambda_2} \cdot \left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right) \cdot Y_1(0) + e^{-\lambda_2 t} \cdot Y_2(0), & \text{whenever } \lambda_1 \neq \lambda_2 \\ -\lambda_{21} t \cdot e^{-\lambda_1 t} Y_1(0) + e^{-\lambda_1 t} \cdot Y_2(0) & \text{otherwise.} \end{cases}$$

Define

$$\begin{aligned} I_1(t) &:= \int_0^t e^{\lambda_1 u} d\tilde{W}_1(u) \\ I_2(t) &:= \int_0^t e^{\lambda_2 u} d\tilde{W}_1(u) \\ I_3(t) &:= \int_0^t e^{\lambda_2 u} d\tilde{W}_2(u) \\ I_4(t) &:= \int_0^t u e^{\lambda_1 u} d\tilde{W}_1(u) \end{aligned}$$

Therefore,

$$Y_1 - \tilde{\mathbb{E}} Y_1(t) = e^{-\lambda t} \cdot I_1(t)$$

Additionally,

$$Y_2 - \tilde{\mathbb{E}} Y_2(t) = \begin{cases} \frac{\lambda_{21}}{\lambda_1 - \lambda_2} \cdot \left(e^{-\lambda_1 t} I_1(t) - e^{-\lambda_2 t} I_2(t)\right) - e^{-\lambda_2 t} I_3(t) & \text{whenever } \lambda_1 \neq \lambda_2 \\ -\lambda_{21} t e^{-\lambda_1 t} I_1(t) + \lambda_{21} e^{-\lambda_1 t} I_4(t) + e^{-\lambda_1 t} I_3(t) & \text{otherwise.} \end{cases}$$

We this can compute the statistics of $Y_1(t)$ and $Y_2(t)$ using only the statistics of $I_i(t)$ for $1 \leq i \leq 4$.

1. Compute the following terms

$$\tilde{\mathbb{E}} I_i(t) I_j(t), \quad \forall 1 \leq i, j \leq 4$$

2. Some derivatives involve time spreads meaning that they depend on interest rate at two different times. As such, we may be interested in computing joint statistics of factor processes at different times. As an example, compute the following term

$$\tilde{\mathbb{E}} I_1(s) I_2(t) \quad 0 \leq s \leq t$$

Proof

1. We answer a more general question. Denote

$$\begin{aligned} I(t) &= \int_0^t t^{a^*} e^{at} \mathrm{d}\tilde{W}_1(t) \\ J(t) &= \int_0^t t^{b^*} e^{bt} \mathrm{d}\tilde{W}_2(t) \\ \mathrm{d}\tilde{W}_1(t) \mathrm{d}\tilde{W}_2(t) &= \rho \mathrm{d}t \end{aligned}$$

Then

$$\begin{aligned}
\mathrm{d}I(t)\mathrm{d}J(t) &= \rho t^{a^*+b^*} e^{(a+b)t}\mathrm{d}t \\
\mathrm{d}I(t)J(t) &= t^{b^*} e^{bt} I(t)\mathrm{d}\tilde{W}_2(t) + t^{a^*} e^{at} J(t)\mathrm{d}\tilde{W}_1(t) + \rho t^{a^*+b^*} e^{(a+b)t}\mathrm{d}t.
\end{aligned}$$

Itô integrals have zero expectations. Thus

$$\tilde{\mathbb{E}}[I(t)J(t)] = \rho \int_0^t t^{a^*+b^*} e^{(a+b)t}\mathrm{d}t$$

Thus

$$\begin{aligned}
\tilde{\mathbb{E}}I_1^2(t) &= \int_0^t e^{2\lambda_1 t}\mathrm{d}t = \frac{1}{2\lambda_1}\left(e^{2\lambda_1 t} - 1\right) \\
\tilde{\mathbb{E}}I_1(t)I_2(t) &= \int_0^t e^{(\lambda_1+\lambda_2)t}\mathrm{d}t = \frac{1}{\lambda_1+\lambda_2}\left(e^{(\lambda_1+\lambda_2)t} - 1\right) \\
\tilde{\mathbb{E}}I_1(t)I_3(t) &= \underbrace{\rho}_{=0} \times \cdots = 0 \\
\tilde{\mathbb{E}}I_1(t)I_4(t) &= \int_0^t u e^{2\lambda_1 u}\mathrm{d}u \\
&= -\frac{1}{2\lambda_1}\int_0^t e^{2\lambda_1 u}\mathrm{d}u + \frac{t}{2\lambda_1}\cdot e^{2\lambda_1 t} \\
&= \frac{1}{4\lambda_1^2}\left(1 - e^{2\lambda_1 t}\right) + \frac{t}{2\lambda_1}\cdot e^{2\lambda_1 t} \\
\tilde{\mathbb{E}}I_4^2(t) &= \int_0^t u^2 e^{2\lambda_1 t}\mathrm{d}t \\
&= \frac{1}{8\lambda_1^3}\left[4\lambda_1^2 t^2 e^{2\lambda_1 t} - 4\lambda_1 t e^{2\lambda_1 t} + 2e^{2\lambda_1 t} - 2\right]
\end{aligned}$$

Last identity, we used integration by parts to obtain

$$\int_0^t u^2 e^{au}\,du = \frac{1}{a^3}(e^{at}(a^2t^2 - 2at + 2) - 2)$$

2. For adapted processes $\Delta_1(t), \Delta_2(t)$ (possibly with jumps) the following is true.

$$\mathbb{E}\int_0^t \Delta_1(s)\mathrm{d}W(s)\int_0^t \Delta_2(s)\mathrm{d}W(s) = \mathbb{E}\int_0^t \Delta_1(s)\Delta_2(s)\mathrm{d}s \qquad \text{(General Itô isometry)}$$

Note that

$$\begin{aligned}
\mathbb{E}\left(\int_0^t (\Delta_1(s)+\Delta_2(s))\,\mathrm{d}W(s)\right)^2 &= \mathbb{E}\int_0^t (\Delta_1(s)+\Delta_2(s))^2\,\mathrm{d}s \\
&= \mathbb{E}\int_0^t \Delta_1^2(s)\mathrm{d}s \\
&\quad + \mathbb{E}\int_0^t \Delta_2^2(s)\mathrm{d}s \\
&\quad + 2\mathbb{E}\int_0^t \Delta_1(s)\Delta_2(s)\mathrm{d}s
\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathbb{E}\left(\int_0^t (\Delta_1(s)+\Delta_2(s))\,\mathrm{d}W(s)\right)^2 &= \mathbb{E}\left(\int_0^t \Delta_1(s)\mathrm{d}W(s)\right)^2 \\ &\quad + \mathbb{E}\left(\int_0^t \Delta_2(s)\mathrm{d}W(s)\right)^2 \\ &\quad + 2\mathbb{E}\int_0^t \Delta_1(s)\mathrm{d}W(s)\int_0^t \Delta_2(s)\mathrm{d}W(s)\end{aligned}$$

Itô isometry yields (General Itô isometry). Note that since $\Delta_i(t)$ are not necessary continuous and they are only required to be L^2-intergrable, it must holds that

$$\mathbb{E}\int_0^{t_1} \Delta_1(s)\mathrm{d}W(s)\int_0^{t_2} \Delta_2(s)\mathrm{d}W(s) = \mathbb{E}\int_0^{t_1\wedge t_2} \Delta_1(s)\Delta_2(s)\mathrm{d}s$$

Just replace $\Delta(u)$ for $0 \le u \le s$ with $\Delta(u)\mathbf{1}_{\{u\le s\}}$ for $0 \le u \le t$. We thus conclude that

$$\tilde{\mathbb{E}}I_1(s)I_2(t) = \tilde{\mathbb{E}}I_1(s)I_2(s) = \frac{1}{\lambda_1+\lambda_2}\left(e^{(\lambda_1+\lambda_2)s}-1\right)$$