Exercise 10.12

Price of a payment of backset LIBOR $L(T,T)$ at time $T+\delta$ at time 0 is computed as below.

$$B(0,T+\delta)L(0,T).$$

Use risk-neutral pricing to prove this result.

Proof

Forward LIBOR rates are computed as follows.

$$1+\delta L(t,T) = \frac{B(t,T)}{B(t,T+\delta)}$$

The desired value is expresses as below.

$$\tilde{\mathbb{E}}\left[D(T+\delta)L(T,T)\right]$$

Remember that

$$B(t,T) = \frac{1}{D(t)}\tilde{\mathbb{E}}\left[D(T)|\mathcal{F}(t)\right]$$

Thus,

$$D(T) = \tilde{\mathbb{E}}\left[\frac{D(T+\delta)}{B(T,T+\delta)}|\mathcal{F}(T)\right] \Rightarrow \tilde{\mathbb{E}}\left[D(T)\right] = \tilde{\mathbb{E}}\left[\frac{D(T+\delta)}{B(T,T+\delta)}\right]$$

On the other hand,

$$B(0,T) = \tilde{\mathbb{E}}\left[D(T)\mathcal{F}(0)\right] = \tilde{\mathbb{E}}\left[D(T)\right]$$

We have that

$$\begin{aligned}
\tilde{\mathbb{E}}\left[D(T+\delta)L(T,T)\right] &= \frac{1}{\delta}\tilde{\mathbb{E}}\left[D(T+\delta)\delta L(T,T)\right] \\
&= \frac{1}{\delta}\tilde{\mathbb{E}}\left[D(T+\delta)\left(\frac{B(T,T)}{B(T,T+\delta)}-1\right)\right] \\
&= \frac{1}{\delta}\tilde{\mathbb{E}}\left[\frac{D(T+\delta)}{B(T,T+\delta)}\right] - \frac{1}{\delta}\tilde{\mathbb{E}}\left[D(T+\delta)\right] \\
&= \frac{1}{\delta}\tilde{\mathbb{E}}\left[D(T)\right] - \frac{1}{\delta}\tilde{\mathbb{E}}\left[D(T+\delta)\right] \\
&= \frac{1}{\delta}\tilde{\mathbb{E}}\left[D(T)\right] - \frac{1}{\delta}B(0,T+\delta) \\
&= \frac{1}{\delta}B(0,T) - \frac{1}{\delta}B(0,T+\delta) \\
&= B(0,T+\delta)\cdot\underbrace{\frac{1}{\delta}\left(\frac{B(0,T)}{B(0,T+\delta)}-1\right)}_{=L(0,T)} \\
&= B(0,T+\delta)L(0,T).
\end{aligned}$$