Exercise 10.2 (ODEs for the mixed affine yield model)

In the mixed model discussed in this chapter, the two factor Vasicek model as well as the two factor CIR model, the price of zero coupon bonds are calculated as follows

$$f(t, y_1, y_2) = e^{-y_1 C_1(T-t) - y_2 C_2(T-t) - A(T-t)},$$

where $C_1(0) = C_2(0) = A(0) = 0$. Find the ordinary differential equations satisfied by A, C_1, C_2.

Proof

The canonical two-factor mixed model is as follows.

$$\begin{aligned} \mathrm{d}Y_1(t) &= (\mu - \lambda_1 Y_1(t))\,\mathrm{d}t + \sqrt{Y_1(t)}\mathrm{d}\tilde{W}_1(t) \\ \mathrm{d}Y_2(t) &= -\lambda_2 Y_2(t)\mathrm{d}t + \sigma_{21}\sqrt{Y_1(t)}\mathrm{d}\tilde{W}_1(t) + \sqrt{\alpha + \beta Y_1(t)}\mathrm{d}\tilde{W}_2(t) \\ R(t) &= \delta_0 + \delta_1 Y_1(t) + \delta_2 Y_2(t). \end{aligned}$$

- $\mathrm{d}D(t) f(t, y_1, y_2)$ is a martingale and thus $\mathrm{d}t$-free. We have that

$$\begin{aligned} \mathrm{d}t\text{-term in } \mathrm{d}f = f_t &+ f_{y_1}(\mu - \lambda_1 y_1) + \tfrac{1}{2} f_{y_1 y_1} y_1 - \lambda_2 f_{y_2} y_2 + \tfrac{1}{2}\sigma_{21}^2 y_1 f_{y_2 y_2} \\ &+ \tfrac{1}{2}(\alpha + \beta y_1) f_{y_2 y_2} + \sigma_{21} f_{y_1 y_2} y_1 \end{aligned}$$

Moreover,

$$\mathrm{d}t\text{-term in } \mathrm{d}Df = -RDf\mathrm{d}t + D\ \mathrm{d}t\text{-term in } \mathrm{d}f$$

Equating this expression to zero and cancelling out the D factor, we obtain

$$\begin{aligned} f_t + f_{y_1}(\mu - \lambda_1 y_1) + \tfrac{1}{2} f_{y_1 y_1} y_1 - \lambda_2 f_{y_2} y_2 + \tfrac{1}{2}\sigma_{21}^2 y_1 f_{y_2 y_2} \\ + \tfrac{1}{2}(\alpha + \beta y_1) f_{y_2 y_2} + \sigma_{21} f_{y_1 y_2} y_1 = Rf \quad \text{(Mixed affine-yield model ODE)} \end{aligned}$$

Let $\tau = T - t$. Derivations of f are as follows:

$$\begin{aligned} f_t &= \left(A'(\tau) + C_2'(\tau) y_2 + C_1'(\tau) y_1\right) \cdot f \\ f_{y_1} &= -C_1(\tau) \cdot f \\ f_{y_2} &= -C_2(\tau) \cdot f \\ f_{y_1 y_1} &= C_1^2(\tau) \cdot f \\ f_{y_2 y_2} &= C_2^2(\tau) \cdot f \\ f_{y_1 y_2} &= C_1(\tau) C_2(\tau) \cdot f \end{aligned}$$

Using these equations and cancelling out factor f from both sides in (Mixed affine-yield model ODE), we obtain that

$$\begin{aligned} A'(\tau) + C_2'(\tau) y_2 + C_1'(\tau) y_1 - C_1(\tau)(\mu - \lambda_1 y_1) \\ + \tfrac{1}{2} C_1^2(\tau) y_1 + \lambda_2 C_2(\tau) y_2 \\ + \tfrac{1}{2}\sigma_{21}^2 y_1 C_2^2(\tau) + \tfrac{1}{2}(\alpha + \beta y_1) C_2^2(\tau) \\ + \sigma_{21} C_1(\tau) C_2(\tau) y_1 = \delta_0 + \delta_1 y_1 + \delta_2 y_2 \end{aligned}$$

Rearranging terms gives

$$\begin{aligned}
&A'(\tau) - C_1(\tau)\mu + \tfrac{\alpha}{2}C_2^2(\tau) \\
&+ y_1\left[C_1'(\tau) + \lambda_1 C_1(\tau) + \tfrac{1}{2}C_1^2(\tau) + \tfrac{1}{2}\sigma_{21}^2 C_2^2(\tau) + \tfrac{\beta}{2}C_2^2(\tau) + \sigma_{21}C_1(\tau)C_2(\tau)\right] \\
&+ y_2\left[C_2'(\tau) + \lambda_2 C_2(\tau)\right] \\
&= \delta_0 + \delta_1 y_1 + \delta_2 y_2
\end{aligned}$$

Since this expression holds for all $y_1, y_2 \geq 0$, it must hold that

$$\begin{aligned}
A'(\tau) - C_1(\tau)\mu + \tfrac{\alpha}{2}C_2^2(\tau) &= \delta_0 \\
C_1'(\tau) + \lambda_1 C_1(\tau) + \tfrac{1}{2}C_1^2(\tau) + \tfrac{1}{2}\sigma_{21}^2 C_2^2(\tau) + \tfrac{\beta}{2}C_2^2(\tau) + \sigma_{21}C_1(\tau)C_2(\tau) &= \delta_1 \\
C_2'(\tau) + \lambda_2 C_2(\tau) &= \delta_2.
\end{aligned}$$