Exercise 10.3 (Calibration of two-factor Vasicek model)

The canonical form of two-factor Vasicek model is as follows.

$$\begin{aligned} \mathrm{d}Y_1(t) &= -\lambda_1 Y_1(t)\mathrm{d}t + \mathrm{d}\tilde{W}_1(t) \\ \mathrm{d}Y_2(t) &= -\lambda_{21} Y_1(t)\mathrm{d}t - \lambda_2 Y_2(t)\mathrm{d}t + \mathrm{d}\tilde{W}_2(t) \\ R(t) &= \delta_0(t) + \delta_1 Y_1(t) + \delta_2 Y_2(t). \end{aligned}$$

Here δ_0 is a nonrandom function of t. Let

$$\begin{aligned} B(t,T) &= \tilde{\mathbb{E}}\left[e^{-\int_t^T R(u)\mathrm{d}u} | \mathcal{F}(t)\right] \\ &= f(t,T,Y_1(t),Y_2(t)). \end{aligned}$$

Here

$$f(t,T,y_1,y_2) = e^{-y_1 C_1(t,T) - y_2 C_2(t,T) - A(t,T)}$$

Assume the model parameters $\lambda_1 > 0, \lambda_2 > 0, \lambda_{12}, \delta_1, \delta_2$ are given. Find $\delta_0(T)$ in terms of $\frac{\partial}{\partial T}\log B(0,T)$ and the model parameters such that

$$f(0,T,Y_1(0),Y_2(0)) = B(0,T), \quad T \geq 0.$$

Proof

Let $\tau = T - t$. From the textbook, the following holds

$$\begin{aligned} \tilde{C}_1'(\tau) &= -\lambda_1 \tilde{C}_1(\tau) - \lambda_{21}(\tau) + \delta_1 \\ \tilde{C}_2'(\tau) &= -\lambda_2 \tilde{C}_2(\tau) + \delta_2 \\ A'(\tau) &= -\tfrac{1}{2}\tilde{C}_1^2(\tau) - \tfrac{1}{2}\tilde{C}_2^2(\tau) + \delta_0(\tau) \end{aligned}$$

Moreover, if $\lambda_1 \neq \lambda_2$, then

$$C_1(t,T) = \tilde{C}_1(\tau) = \frac{1}{\lambda_1}\left(\delta_1 - \frac{\lambda_{21}\delta_2}{\lambda_2}\right)\left(1 - e^{-\lambda_1\tau}\right) + \frac{\lambda_{21}\delta_2}{\lambda_2(\lambda_1 - \lambda_2)}\left(e^{-\lambda_2\tau} - e^{-\lambda_1\tau}\right)$$

And if $\lambda_1 = \lambda_2$, then

$$C_1(t,T) = \tilde{C}_1(\tau) = \frac{1}{\lambda_1}\left(\delta_1 - \frac{\lambda_{21}\delta_1}{\lambda_2}\right)\left(1 - e^{-\lambda_1\tau}\right) + \frac{\lambda_{21}\delta_2}{\lambda_1}\tau e^{-\lambda_1\tau}$$

Also

$$C_2(t,T) = \tilde{C}_2(\tau) = \frac{\delta_2}{\lambda_2}\left(1 - e^{-\lambda_2\tau}\right)$$

Finally, from (10.2.55),

$$A(t,T) = A(\tau) = \int_0^\tau \left[-\frac{1}{2}C_1^2(u) - \frac{1}{2}C_2^2(u) + \delta_0(t+u)\right]\mathrm{d}u$$

Therefore,

$$A(0,T) = \int_0^T \left[-\frac{1}{2}C_1^2(u,T) - \frac{1}{2}C_2^2(u,T) + \delta_0(u)\right]\mathrm{d}u$$

So

$$\frac{\partial}{\partial T}A(0,T) = -\frac{1}{2}C_1^2(T,T) - \frac{1}{2}C_2^2(T,T) + \delta_0(T)$$
$$= \delta_0(T).$$

On the other hand,

$$\log B(0,T) = -Y_1(0)C_1(0,T) - Y_2(0)C_2(0,T) - A(0,T)$$

Thus,

$$\delta_0(T) = -\frac{\partial}{\partial T}\log B(0,T) - Y_1(0)\frac{\partial}{\partial T}C_1(0,T) - Y_2(0)\frac{\partial}{\partial T}C_2(0,T)$$

Computing $C_1(0,T)$ and $C_2(0,T)$ using their closed-form formulas as above completes the proof.