Exercise 10.4 (Original Hull-White two-factor model)

Hull-White two-factor model was originally defined as

$$\begin{aligned}
\mathrm{d}U(t) &= -\lambda_1 U(t)\mathrm{d}t + \sigma_1 \mathrm{d}\tilde{B}_2(t) \\
\mathrm{d}R(t) &= [\theta(t) + U(t) - \lambda_2 R(t)]\,\mathrm{d}t + \sigma_2 \mathrm{d}\tilde{B}_1(t).
\end{aligned}$$

Here $\lambda_1, \lambda_2, \sigma_1, \sigma_2$ are positive constants. $\theta(t)$ is non-random function and $\mathrm{d}\tilde{B}_1(t)\mathrm{d}\tilde{B}_2(t) = \rho\mathrm{d}t$. Derive a formula for $R(t)$ as below

$$R(t) = \delta_0(t) + \delta_1 Y_1(t) + \delta_2 Y_2(t).$$

Here canonical two-factor Vasicek equations hold in vector form as follows

$$\mathrm{d}Y(t) = -\Lambda Y(t) + \mathrm{d}\tilde{W}(t).$$

Proof

Denote

$$\begin{aligned}
X(t) &= \begin{bmatrix} U(t) \\ R(t) \end{bmatrix} \\
K &= \begin{bmatrix} \lambda_1 & 0 \\ -1 & \lambda_2 \end{bmatrix} \\
\Sigma &= \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \\
\Theta(t) &= \begin{bmatrix} 0 \\ \theta(t) \end{bmatrix} \\
\tilde{B}(t) &= \begin{bmatrix} \tilde{B}_1(t) \\ \tilde{B}_2(t) \end{bmatrix}
\end{aligned}$$

Then

$$\mathrm{d}X(t) = \Theta(t)\mathrm{d}t - KX(t)\mathrm{d}t + \Sigma\mathrm{d}\tilde{B}(t).$$

Denote

$$\hat{X}(t) = X(t) - e^{-Kt}\int_0^t e^{Ku}\Theta(u)\mathrm{d}u.$$

We have that

$$\begin{aligned}
\mathrm{d}\hat{X}(t) &= \mathrm{d}X(t) + Ke^{-Kt}\int_0^t e^{Ku}\Theta(u)\mathrm{d}u\mathrm{d}t - \Theta(t)\mathrm{d}t \\
&= -KX(t)\mathrm{d}t + \Sigma\mathrm{d}\tilde{B}(t) + Ke^{-Kt}\int_0^t e^{Ku}\Theta(u)\mathrm{d}u\mathrm{d}t \\
&= -K\left[X(t)\mathrm{d}t - e^{-Kt}\int_0^t e^{Ku}\Theta(u)\mathrm{d}u\right]\mathrm{d}t + \Sigma\mathrm{d}\tilde{B}(t) \\
&= -K\hat{X}(t) + \Sigma\mathrm{d}\tilde{B}(t).
\end{aligned}$$

Next, we find a matrix C such that

$$\tilde{W}(t) := C\Sigma\tilde{B}(t)$$

contains two independent Brownian motions $\tilde{W}_1(t)$ and $\tilde{W}_2(t)$. Since $\tilde{B}_i(t)$ is a martingale and linear combinations of martingales are martingales, we conclude that for any 2×2 matrix C, $\tilde{W}_i(t)$ is martingale for $i = 1, 2$. According to Levy theorem, it suffices to ensure that

$$\mathrm{d}\tilde{W}_i(t)\mathrm{d}\tilde{W}_j(t) = \delta_{i,j}\mathrm{d}t.$$

Rewriting, this is equivalent to

$$\mathrm{d}\tilde{W}(t)\mathrm{d}\tilde{W}(t)^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathrm{d}t$$

Notice

$$\begin{aligned} \mathrm{d}\tilde{W}(t)\mathrm{d}\tilde{W}(t)^T &= C\Sigma\mathrm{d}\tilde{B}(t)\mathrm{d}\tilde{B}(t)^T\Sigma^T C^T \\ &= C\Sigma \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \Sigma^T C^T \mathrm{d}t \end{aligned}$$

Let

$$C = \begin{bmatrix} \frac{1}{\sigma_1} & 0 \\ -\frac{\rho}{\sigma_1\sqrt{1-\rho^2}} & \frac{1}{\sigma_2\sqrt{1-\rho^2}} \end{bmatrix}$$

Then

$$C\Sigma = \begin{bmatrix} 1 & 0 \\ -\frac{\rho}{\sqrt{1-\rho^2}} & \frac{1}{\sqrt{1-\rho^2}} \end{bmatrix}$$

Continuing,

$$\begin{aligned} C\Sigma \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \Sigma^T C^T \mathrm{d}t &= \begin{bmatrix} 1 & 0 \\ -\frac{\rho}{\sqrt{1-\rho^2}} & \frac{1}{\sqrt{1-\rho^2}} \end{bmatrix} \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{\rho}{\sqrt{1-\rho^2}} \\ 0 & \frac{1}{\sqrt{1-\rho^2}} \end{bmatrix} \mathrm{d}t \\ &= \begin{bmatrix} 1 & \rho \\ 0 & \sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} 1 & -\frac{\rho}{\sqrt{1-\rho^2}} \\ 0 & \frac{1}{\sqrt{1-\rho^2}} \end{bmatrix} \mathrm{d}t \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathrm{d}t. \end{aligned}$$

Denote

$$Y(t) = C\hat{X}(t).$$

Then

$$\begin{aligned} \mathrm{d}Y(t) &= -\underbrace{CKC^{-1}C}_{:=\Lambda}\hat{X}(t) + C\Sigma\mathrm{d}\tilde{B}(t) \\ &= -\Lambda Y(t) + \mathrm{d}\tilde{W}(t). \end{aligned}$$

Denote

$$[\delta_1, \delta_2] := [0, 1]C^{-1}.$$

Then

$$
\begin{aligned}
R(t) - \underbrace{e^{-Kt}\int_0^t e^{Ku}\theta(u)\mathrm{d}u}_{:=\delta_0(t)} &= [0,1]\hat{X}(t) \\
&= [\delta_1, \delta_2]C\hat{X}(t) \\
&= [\delta_1, \delta_2]Y(t)
\end{aligned}
$$

Rewrite to get

$$
R(t) = \delta_0(t) + \delta_1 Y_1(t) + \delta_2 Y_2(t).
$$