Exercise 10.6 (Degenerate two-factor Vasicek model)

The canonical form of two-factor Vasicek model is as follows.

$$\begin{aligned}
\mathrm{d}Y_1(t) &= -\lambda_1 Y_1(t)\mathrm{d}t + \mathrm{d}\tilde{W}_1(t) \\
\mathrm{d}Y_2(t) &= -\lambda_{21} Y_1(t)\mathrm{d}t - \lambda_2 Y_2(t)\mathrm{d}t + \mathrm{d}\tilde{W}_2(t) \\
R(t) &= \delta_0 + \delta_1 Y_1(t) + \delta_2 Y_2(t).
\end{aligned}$$

Prove that in either of the following cases, this model will be be reduced to a one dimensional SDE for the interest rate $R(t)$.

1. $\delta_2 = 0$
2. $(\lambda_1 - \lambda_2)\delta_1 + \lambda_{21}\delta_2 = 0$ and $\delta_i > 0$ for $i = 1, 2, 3$.

Proof

If $\delta_2 = 0$, then the following one-dimensional SDE holds

$$\begin{aligned}
R(t) &= \delta_0 + \delta_2 Y_1(t) \\
\mathrm{d}Y_1(t) &= -\lambda_1 Y_1(t)\mathrm{d}t + \mathrm{d}\tilde{W}_1(t)
\end{aligned}$$

Now assume that $\delta_i > 0$ and $(\lambda_1 - \lambda_2)\delta_1 + \lambda_{21}\delta_2 = 0$. We have

$$\begin{aligned}
\mathrm{d}R(t) &= \delta_1 \left[-\lambda_1 Y_1(t)\mathrm{d}t + \mathrm{d}\tilde{W}_1(t) \right] + \delta_2 \left[-\lambda_{21} Y_1(t)\mathrm{d}t - \lambda_2 Y_2(t)\mathrm{d}t + \mathrm{d}\tilde{W}_2(t) \right] \\
&= Y_1(t) \left[-\delta_1\lambda_1 - \delta_2\lambda_{21} \right] \mathrm{d}t - \delta_2\lambda_2 Y_2(t)\mathrm{d}t + \delta_1 \mathrm{d}\tilde{W}_1(t) + \delta_2 \mathrm{d}\tilde{W}_2(t) \\
&= Y_1(t) \left[-\lambda_2\delta_1 \right] \mathrm{d}t - \delta_2\lambda_2 Y_2(t)\mathrm{d}t + \delta_1 \mathrm{d}\tilde{W}_1(t) + \delta_2 \mathrm{d}\tilde{W}_2(t) \\
&= -\lambda_2 \left[\delta_1 Y_1(t) + \delta_2 Y_2(t) \right] \mathrm{d}t + \delta_1 \mathrm{d}\tilde{W}_1(t) + \delta_2 \mathrm{d}\tilde{W}_2(t).
\end{aligned}$$

Thus, we need

$$\begin{aligned}
-\lambda_2 \left[\delta_1 Y_1(t) + \delta_2 Y_2(t) \right] &= a - bR(t) \\
\delta_1 \mathrm{d}\tilde{W}_1(t) + \delta_2 \mathrm{d}\tilde{W}_2(t) &= \sigma \mathrm{d}\tilde{B}(t)
\end{aligned}$$

With regards to the second equation, simply write

$$\tilde{B}(t) = \frac{\delta_1}{\sqrt{\delta_1^2 + \delta_2^2}} \tilde{W}_1(t) + \frac{\delta_2}{\sqrt{\delta_1^2 + \delta_2^2}} \tilde{W}_2(t)$$

Clearly, $\tilde{B}(t)$ is a martingale. Moreover,

$$\mathrm{d}\tilde{B}(t)\mathrm{d}\tilde{B}(t) = \frac{\delta_1^2}{\delta_1^2 + \delta_2^2}\mathrm{d}t + \frac{\delta_2^2}{\delta_1^2 + \delta_2^2}\mathrm{d}t = \mathrm{d}t.$$

Thus, by Levy theorem, $\tilde{B}(t)$ is a Brownian motion. Letting $\sigma = \sqrt{\delta_1^2 + \delta_2^2}$, the second equation follows. Finally,

$$a - bR(t) = (a - b\delta_0) - b\delta_1 Y_1(t) - b\delta_2 Y_2(t)$$

Thus to satisfy the first equation, it suffices to have

$$
\begin{aligned}
-\lambda_2\delta_1 &= -b\delta_1 \\
-\lambda_2\delta_2 &= -b\delta_2 \\
a - b\delta_0 &= 0
\end{aligned}
$$

So let

$$
\begin{aligned}
b &= \lambda_2 \\
a &= \lambda_2\delta_0
\end{aligned}
$$