Exercise 10.7 (Forward measure in the two-factor Vasicek model)

Proof

The risk-neutral pricing formula in terms of T-forward measure is as the following.

$$V(t) = \underbrace{B(t,T)}_{:=f(t,Y_1(t),Y_2(t))} \quad \tilde{\mathbb{E}}^T[V(T)|\mathcal{F}(t)]$$

Therefore, since the bond's price at time T is equal to

$$B(T,\overline{T}) = e^{-C_1(\overline{T}-T)Y_1(T)-C_2(\overline{T}-T)Y_2(T)-A(\overline{T}-T)}$$

Thus, the price of the call at expiration T is equal to

$$V(T) = \left(\exp\left(\underbrace{-C_1(\overline{T}-T)Y_1(T) - C_2(\overline{T}-T)Y_2(T) - A(\overline{T}-T)}_{:=X}\right) - K\right)^+.$$

It then immediately follows that

$$V(t) = B(t,T)\tilde{\mathbb{E}}^T\left[\left(e^X - K\right)^+ |\mathcal{F}(t)\right]$$

Next, we have that

$$\begin{aligned}
\mathrm{d}D(t)B(t,T) &= [\cdots]\mathrm{d}t + D\left[f_{y_1}\mathrm{d}\tilde{W}_1(t) + f_{y_2}\mathrm{d}\tilde{W}_2(t)\right] \\
&= D(t)\left[f_{y_1}\mathrm{d}\tilde{W}_1(t) + f_{y_2}\mathrm{d}\tilde{W}_2(t)\right] \\
&= D(t)f(t,Y_1(t),Y_2(t))\left[-C_1(T-t)\mathrm{d}\tilde{W}_1(t) - C_2(T-t)\mathrm{d}\tilde{W}_2(t)\right]
\end{aligned}$$

According to Girsanov theorem or (9.2.6), $\tilde{W}_1^T(t), \tilde{W}_2^T(t)$ are Brownian motions under $\tilde{\mathbb{P}}^T$ where

$$\tilde{W}_j^T(t) = \int_0^t C_j(T-u)\mathrm{d}u + \tilde{W}_j(t), \quad j = 1,2.$$

The two-factro Vasicek model is as follows

$$\mathrm{d}Y(t) = -\Lambda Y(t) + \mathrm{d}\tilde{W}(t).$$

This will lead to

$$\begin{aligned}
Y(t) &= e^{-\Lambda t}Y(0) + \int_0^t e^{-\Lambda(t-u)}\mathrm{d}\tilde{W}(u) \\
&= e^{-\Lambda t}Y(0) + \int_0^t e^{-\Lambda(t-u)}\left[\mathrm{d}\tilde{W}^T(u) - [\text{some deterministic func. of u}]\mathrm{d}u\right] \\
&= e^{-\Lambda t}Y(0) + \int_0^t [\text{some deterministic func. of u}]\mathrm{d}u + \int_0^t e^{-\Lambda(t-u)}\mathrm{d}\tilde{W}^T(u)
\end{aligned}$$

Thus, $Y_1(T), Y_2(T)$ are both normally distributed under $\tilde{\mathbb{P}}^T$ and since C_i are deterministic functions, we conclude that X must be normally distributed under $\tilde{\mathbb{P}}^T$ as well. Denote

$$X = \mu - \frac{1}{2}\sigma^2 - \sigma Z$$

where Z is standard normal under $\tilde{\mathbb{P}}^T$. We have

$$X \geq \log K \iff Z \leq \underbrace{\frac{1}{\sigma}\left(\mu - \tfrac{1}{2}\sigma^2 - \log K\right)}_{:=d_-}$$

$$\begin{aligned}
V(0) &= B(0,T)\tilde{\mathbb{E}}^T\left[(e^X - K)^+\right] \\
&= B(0,T)\tilde{\mathbb{E}}^T\left[e^X \mathbf{1}_{\{Z \leq d_-\}} - K\mathbf{1}_{\{Z \leq d_-\}}\right] \\
&= B(0,T)\tilde{\mathbb{E}}^T\left[e^X \mathbf{1}_{\{Z \leq d_-\}}\right] - KB(0,T)\tilde{\mathbb{E}}^T\left[\mathbf{1}_{\{Z \leq d_-\}}\right] \\
&= B(0,T)\tilde{\mathbb{E}}^T\left[e^X \mathbf{1}_{\{Z \leq d_-\}}\right] - KB(0,T)N(d_-).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\tilde{\mathbb{E}}^T\left[e^X \mathbf{1}_{\{Z \leq d_-\}}\right] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-} e^{\mu - \frac{1}{2}\sigma^2 - \sigma z} e^{-\frac{z^2}{2}}\,\mathrm{d}z \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_- + \sigma} e^{\mu - \frac{1}{2}\sigma^2 - \sigma(u-\sigma)} e^{-\frac{(u-\sigma)^2}{2}}\,\mathrm{d}u \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_- + \sigma} e^{-\frac{u^2}{2}}\,\mathrm{d}u \\
&= N(\underbrace{d_- + \sigma}_{:=d_+}).
\end{aligned}$$

Putting pieces together, we obtain that

$$V(0) = B(0,T)e^{\mu}N(d_+) - KB(0,T)N(d_+).$$