Exercise 10.9 (Multifactor HJM model)

Proof

We begin by noting that

$$\begin{aligned}
\mathrm{d} - \int_t^T f(t,v)\mathrm{d}v &= f(t,t)\mathrm{d}t - \int_t^T \mathrm{d}f(t,v)\mathrm{d}v \\
&= R(t)\mathrm{d}t - \int_t^T \left[\alpha(t,v)\mathrm{d}t + \sum_{j=1}^d \sigma_j(t,v)\mathrm{d}W_j(t)\right]\mathrm{d}v \\
&= R(t)\mathrm{d}t - \underbrace{\int_t^T \alpha(t,v)\mathrm{d}v}_{:=\alpha^*(t,T)}\mathrm{d}t - \sum_{j=1}^d \underbrace{\int_t^T \sigma_j(t,v)\mathrm{d}v}_{:=\sigma_j^*(t,T)}\mathrm{d}W_j(t) \\
&= R(t)\mathrm{d}t - \alpha^*(t,T)\mathrm{d}t - \sum_{j=1}^d \sigma_j^*(t,T)\mathrm{d}W_j(t).
\end{aligned}$$

Continuing,

$$B(t,T) = \exp\left(-\int_t^T f(t,v)\mathrm{d}v\right), \quad 0 \le t \le T \le \overline{T}.$$

Therefore,

$$\begin{aligned}
\mathrm{d}B(t,T) = B(t,T)&\left[R(t)\mathrm{d}t - \alpha^*(t,T)\mathrm{d}t - \sum_{j=1}^d \sigma_j^*(t,T)\mathrm{d}W_j(t)\right] \\
&+ \frac{1}{2}B(t,T)\left[\sum_{j=1}^d \sigma_j^*(t,T)^2\right]\mathrm{d}t
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{d}D(t)B(t,T) &= D(t)B(t,T)\left[\left(-\alpha^*(t,T) + \frac{1}{2}\sum_{j=1}^d \sigma_j^*(t,T)^2\right)\mathrm{d}t - \sum_{j=1}^d \sigma_j^*(t,T)\mathrm{d}W_j(t)\right] \\
&= -D(t)B(t,T)\sum_{j=1}^d \sigma_j^*(t,T)\left[\Theta_j(t)\mathrm{d}t + \mathrm{d}W_j(t)\right]
\end{aligned}$$

Thus, we need

$$-\alpha^*(t,T) + \frac{1}{2}\sum_{j=1}^d \sigma_j^*(t,T)^2 = -\sum_{j=1}^d \sigma_j^*(t,T)\Theta_j(t)$$

Taking derivative with respect to T,

$$-\alpha(t,T) + \sum_{j=1}^d \sigma_j(t,T)\sigma_j^*(t,T) = -\sum_{j=1}^d \sigma_j(t,T)\Theta_j(t)$$

In other words,

$$\alpha(t,T) = \sum_{j=1}^{d} \sigma_j(t,T) \left[\Theta_j(t) + \sigma_j^*(t,T)\right]$$

Now suppose that for maturities $T_1, \cdots, T_d$, the matrix $\Sigma(t) := (\sigma_j(t,T_k))_{1\leq j,k\leq d}$ is non-singular. Here $\Sigma(t)_{k,j} := \sigma_j(t,T_k)$. We want to show that $\Theta_j(t)$ is unique for all $1 \leq j \leq d$. Suppose that $\Theta_j^1(t)$ and $\Theta_j^2(t)$ satisfy these equations. Therefore,

$$\sum_{j=1}^{d} \sigma_j(t,T)\Theta_j^1(t) = \sum_{j=1}^{d} \sigma_j(t,T)\Theta_j^2(t)$$

In particular,

$$\sum_{j=1}^{d} \sigma_j(t,T_k) \left[\Theta_j^1(t) - \Theta_j^2(t)\right] = 0 \quad \forall k \in [1,d].$$

Therefore,

$$\Sigma(t)\mathbf{\Theta}(t) = 0 \text{ where } \mathbf{\Theta}(t)_j := \Theta_j^1(t) - \Theta_j^2(t).$$

Since $\Sigma(t)$ is non-singular, it must hold that $\mathbf{\Theta}(t) \equiv \mathbf{0}$. We thus showed that if for any t, $\Sigma(t)$ is non-singular, then $\Theta(t)$ is unique.