Exercise 1.10

$\mathbb{P}$ is the uniform measure on $[0,1]$. Define

$$Z(\omega) = \begin{cases} 0 & \text{if } 0 \leq \omega \leq \frac{1}{2} \\ 2 & \text{if } \frac{1}{2} \leq \omega \leq 1. \end{cases}$$

For each $A \in \mathcal{B}[0,1]$, let

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega) \mathrm{d}\mathbb{P}(\omega).$$

1. Show that $\tilde{\mathbb{P}}$ is a probability measure.
2. Show that $\tilde{\mathbb{P}}$ is absolutely continuous with respect to $\mathbb{P}$ *i.e.*, $\mathbb{P}(A) = 0$ implies $\tilde{\mathbb{P}}(A) = 0$.
3. Show that $\mathbb{P}$ is not absolutely continuous with respect to $\tilde{\mathbb{P}}$.

Proof

1. Since $Z(\omega) \geq 0$ for all $\omega \in [0,1]$, $\tilde{\mathbb{P}}(A) \geq 0$ for every each $A \in \mathcal{B}[0,1]$. Moreover,

$$\begin{aligned} \tilde{\mathbb{P}}([0,1]) &= \int_0^1 Z(\omega) \mathrm{d}\mathbb{P}(\omega) \\ &= 0 \cdot \int_0^{\frac{1}{2}} \mathrm{d}\mathbb{P}(\omega) + 2 \cdot \int_{\frac{1}{2}}^1 \mathrm{d}\mathbb{P}(\omega) \\ &= 1. \end{aligned}$$

2. Since $Z \leq 2$, it holds that

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega) \mathrm{d}\mathbb{P}(\omega) \leq 2 \cdot \int_A \mathrm{d}\mathbb{P}(\omega) = 2 \cdot \mathbb{P}(A).$$

3. $\tilde{\mathbb{P}}\left([0, \frac{1}{2}]\right) = 0 \neq \frac{1}{2} = \mathbb{P}\left([0, \frac{1}{2}]\right)$.