Exercise 1.11

Let X be a standard normal distribution on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $Y = X + \theta$ and denote

$$Z = \exp\left(-\theta X - \frac{\theta^2}{2}\right)$$

Denote by $\tilde{\mathbb{P}}(A) = \int_A Z \mathrm{d}\mathbb{P}$. Verify that

$$\tilde{\mathbb{E}} e^{tY} = e^{\frac{t^2}{2}} \text{ for all } t \in \mathbb{R}.$$

This is an alternative way of showing that Y is standard normal under $\tilde{\mathbb{P}}$.

Proof

Note that

$$\begin{aligned}
\tilde{\mathbb{E}} e^{tY} &= \int_\Omega e^{tY(\omega)} \mathrm{d}\tilde{\mathbb{P}}(\omega) \\
&= \int_\Omega Z(\omega) e^{tY(\omega)} \mathrm{d}\mathbb{P}(\omega) \\
&= \int_\Omega Z(\omega) e^{tX(\omega)+t\theta} \mathrm{d}\mathbb{P}(\omega) \\
&= \int_\Omega e^{tX(\omega)+t\theta-\theta X(\omega)-\frac{\theta^2}{2}} \mathrm{d}\mathbb{P}(\omega) \\
&= e^{t\theta-\frac{\theta^2}{2}} \int_\Omega e^{tX(\omega)-\theta X(\omega)} \mathrm{d}\mathbb{P}(\omega) \\
&= e^{t\theta-\frac{\theta^2}{2}} \mathbb{E} e^{(t-\theta)X} \\
&= e^{t\theta-\frac{\theta^2}{2}} \cdot e^{\frac{(t-\theta)^2}{2}} \\
&= e^{\frac{t^2}{2}}
\end{aligned}$$