Exercise 1.12

Let X be a standard normal random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Set $Y = X + \theta$ and denote $Z = e^{-\theta X - \frac{1}{2}\theta^2}$. Define

$$\tilde{\mathbb{P}}(A) := \int_A Z(\omega) \mathrm{d}\mathbb{P}(\omega).$$

Under $\tilde{\mathbb{P}}$, Y is a standard normal random variable. Note that $X = Y - \theta$. Denote $\hat{Z} = e^{\theta Y - \frac{1}{2}\theta^2}$ and define

$$\hat{\mathbb{P}}(A) := \int_A \hat{Z}(\omega) \mathrm{d}\tilde{\mathbb{P}}(\omega).$$

Show that $\hat{Z} = \frac{1}{Z}$ and $\hat{\mathbb{P}} = \mathbb{P}$.

Proof

We begin by noting that

$$\begin{aligned}
\hat{Z} &= e^{\theta Y - \frac{1}{2}\theta^2} \\
&= e^{\theta(X+\theta) - \frac{1}{2}\theta^2} \\
&= e^{\theta X + \frac{1}{2}\theta^2} \\
&= \frac{1}{Z}.
\end{aligned}$$

On the other hand, we know that

$$\begin{aligned}
\mathbb{P}(A) &= \int_A \frac{1}{Z(\omega)} \mathrm{d}\tilde{\mathbb{P}}(\omega) \\
&= \int_A \hat{Z}(\omega) \mathrm{d}\tilde{\mathbb{P}}(\omega) \\
&= \hat{\mathbb{P}}(A).
\end{aligned}$$