Exercise 1.13 (Change of measure for a normal random variable)

Let X be a standard normal random variable on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $Y = X + \theta$. Show that if Y is standard normal under some probability distribution $\tilde{\mathbb{P}}$, then for 'small' set A and fixed $\omega^* \in \Omega$ s.t. $\omega^* \in A$, the following must hold

$$\frac{\tilde{\mathbb{P}}(A)}{\mathbb{P}(A)} = \exp\left(-\theta X(\omega^*) - \frac{\theta^2}{2}\right).$$

Proof

We begin by showing that

$$\tfrac{1}{\epsilon}\mathbb{P}(X \in B(X(\omega^*), \epsilon)) \approx \tfrac{1}{\sqrt{2\pi}} \exp\left(-\frac{X^2(\omega^*)}{2}\right)$$

Here $B(X(\omega^*), \epsilon)) = [X(\omega^*) - \frac{\epsilon}{2}, X(\omega^*) + \frac{\epsilon}{2}]$. Denote $x = X(\omega^*)$. We have that

$$\frac{1}{\epsilon}\mathbb{P}(X \in B(x, \epsilon)) = \frac{1}{\epsilon\sqrt{2\pi}} \int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} \underbrace{e^{-\frac{t^2}{2}}}_{\approx \exp(-\frac{x^2}{2})} \,\mathrm{d}t \approx \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{x^2}{2}).$$

Therefore, assuming that Y is standard normal under $\tilde{\mathbb{P}}$, we must have that

$$\tfrac{1}{\epsilon}\tilde{\mathbb{P}}(Y \in B(Y(\omega^*), \epsilon)) \approx \tfrac{1}{\sqrt{2\pi}} \exp\left(-\frac{Y^2(\omega^*)}{2}\right)$$

Clearly, $B(Y(\omega^*), \epsilon)) = B(X(\omega^*), \epsilon))$. Therefore,

$$\begin{aligned}
\frac{\tilde{\mathbb{P}}(Y \in B(Y(\omega^*), \epsilon))}{\mathbb{P}(X \in B(x, \epsilon))} &\approx \exp\left(-\frac{1}{2}\left(Y(\omega^*) - X(\omega^*)\right) \cdot \left(Y(\omega^*) + X(\omega^*)\right)\right) \\
&\approx \exp\left(-\frac{\theta}{2} \cdot (\theta + 2X(\omega^*))\right) \\
&= \exp\left(-\frac{\theta^2}{2} - \theta X(\omega^*)\right).
\end{aligned}$$