Exercise 1.14 (Change of measure for an exponential random variable)

Let X be a non-negative random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the exponential distribution. In other words,

$$\mathbb{P}(X \leq a) = 1 - e^{-\lambda a},\ a \geq 0.$$

Let $\tilde{\lambda}$ be another positive constant. Define

$$Z = \frac{\tilde{\lambda}}{\lambda} e^{-(\tilde{\lambda}-\lambda)X}$$

Define $\tilde{\mathbb{P}}$ by

$$\tilde{\mathbb{P}}(A) = \int_A Z \mathrm{d}\mathbb{P}.$$

Show that $\tilde{\mathbb{P}}(\Omega) = 1$. Moreover, find the distribution of X under $\tilde{\mathbb{P}}$.

Proof

We have that

$$\tilde{\mathbb{P}}(X \leq a) = \int_0^a \frac{\tilde{\lambda}}{\lambda} e^{-(\tilde{\lambda}-\lambda)x} \cdot \left(\lambda e^{-\lambda x}\right) \mathrm{d}x = \int_0^a \tilde{\lambda} e^{-\tilde{\lambda} x} \mathrm{d}x = -e^{-\tilde{\lambda} x}|_0^a = 1 - e^{-\tilde{\lambda} a}.$$

Letting $a \to +\infty$, we obtain that $\tilde{\mathbb{P}}(\Omega) = 1$. Moreover, X remains exponential under $\tilde{\mathbb{P}}$; only its parameter changes from λ to $\tilde{\lambda}$.