Exercise 1.15

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random variable X on it with density function $f(x)$ with $f(x) > 0, \forall x \in \mathbb{R}$. Let g be a differentiable function such that

$$g'(t) > 0, \forall t \quad \text{and} \quad \lim_{y \to -\infty} g(y) = -\infty, \lim_{y \to +\infty} g(y) = +\infty.$$

Let $Y = g(X)$ and assume h is an arbitrary probability density function. The main goal of this exercise is show that we could change the probability measure such that h becomes the probability density function of Y. To this end, define

$$Z := \frac{h(g(X))g'(X)}{f(X)} \quad \text{and} \quad \tilde{\mathbb{P}}(A) := \int_A Z \mathrm{d}\mathbb{P} \; \forall A \in \mathcal{F}.$$

Prove that

1. Z is non-negative and $\mathbb{E}Z = 1$
2. Y has density h under $\tilde{\mathbb{P}}$.

Proof

1. Since $h, g', f \geq 0$, we conclude that $Z \geq 0$. Also,

$$\begin{aligned}
\mathbb{E}Z &= \int_{-\infty}^{+\infty} \frac{h(g(x))g'(x)}{f(x)} f(x)dx \\
&= \int_{-\infty}^{+\infty} h(g(x))g'(x)dx \\
&= \int_{-\infty}^{+\infty} h(y)dy \\
&= 1.
\end{aligned}$$

   Here change of variables $y \mapsto g(x)$ and the favorable properties of g are used.

2. For every Borel subset B of $\mathbb{R}$, it holds that

$$\begin{aligned}
\tilde{\mu}_Y(B) &:= \tilde{\mathbb{P}}(Y \in B) \\
&= \int_\Omega \mathbf{1}_B(Y) \mathrm{d}\tilde{\mathbb{P}} \\
&= \int_\Omega \mathbf{1}_B(g(X(\omega))) Z(\omega) \mathrm{d}\mathbb{P}(\omega) \\
&= \int_{\mathbb{R}} \mathbf{1}_B(g(x)) \frac{h(g(x))g'(x)}{f(x)} f(x) \mathrm{d}x \\
&= \int_B h(y) \mathrm{d}y
\end{aligned}$$