Exercise 1.2

Consider the infinite coin-toss space Ω_∞. $A \subseteq \Omega_{+\infty}$ is defined by

$$A := \{(\omega_1, \omega_2, \cdots) \in \Omega_\infty : \omega_1 = \omega_2, \omega_3 = \omega_4, \omega_5 = \omega_6, \cdots\}$$

Show that A is unaccountably infinite. Moreover, show that if $p \in (0,1)$ then $\mathbb{P}(A) = 0$.

Proof

For each $\boldsymbol{\omega} = (\omega_1, \omega_2, \cdots) \in \Omega_\infty$ define

$$\phi(\boldsymbol{\omega}) = (\omega_1, \omega_1, \omega_2, \omega_2, \omega_3, \omega_3, \cdots) \in A.$$

Note that $\phi : \Omega_\infty \to A$ is one-to-one. Therefore, since Ω_∞ is uncountable, so is A. To see that $\mathbb{P}(A) = 0$ for $0 < p < 1$, define

$$A_n := \{(\omega_1, \omega_2, \cdots) \in \Omega_\infty : \omega_1 = \omega_2, \cdots, \omega_{2n-1} = \omega_{2n}\}$$

Then, $A \subseteq A_n$ for each n. Also, $\mathbb{P}(A_n) = \left(p^2 + (1-p)^2\right)^n < 1$. Here we used $p \notin \{0, 1\}$. As $\lim_{n\to+\infty} \mathbb{P}(A_n) = 0$, we conclude that $\mathbb{P}(A) = 0$.