Exercise 1.3

Let $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], 2^{[0,1]}, \mathbb{P})$ where $\mathbb{P}(A) = +\infty$ if A is infinite, otherwise $\mathbb{P}(A) = 0$. Show that $\mathbb{P}(\emptyset) = 0$ and if $A_1, \cdots, A_N$ are disjoint then $\mathbb{P}(\cup_i^n A_i) = \sum_{i=1}^n \mathbb{P}(A_i)$. However, show that if $A_1, \cdots,$ is an infinite sequence of disjoint sets, then $\mathbb{P}(\cup_i^\infty A_i) = \sum_{i=1}^\infty \mathbb{P}(A_i)$ does not necessarily hold.

Proof

Clearly, $\mathbb{P}(\emptyset) = 0$. To see $\mathbb{P}(\cup_i^n A_i) = \sum_{i=1}^n \mathbb{P}(A_i)$, note that either side is 0 if and only if each A_i is finite and is $+\infty$ if and only if there exists i s.t. A_i is infinite. Therefore the equality holds. Now consider an infinite sequence $x_1 < \cdots \in [0,1]$ and define $A_n = \{x_1, \cdots, x_n\}$. We have that

$$+\infty = \mathbb{P}(\cup_i^\infty A_i) \neq \sum_{i=1}^\infty \mathbb{P}(A_i) = 0.$$