Exercise 1.4

Construct a normally distributed random variable Z on the infinite coin-toss probability space $(\Omega_\infty, \mathcal{F}_\infty, \mathbb{P}_\infty)$. Define a sequence of random variables Z_n such that

$$\lim Z_n(\omega) = Z(\omega) \text{ for each } \omega \in \Omega_\infty.$$

Proof

For $\omega \in \Omega_\infty$, denote

$$Y_n(\omega) = \begin{cases} 1 & \text{if } \omega_n = H \\ 0 & \text{if } \omega_n = T \end{cases}$$

Denote

$$Z = N^{-1}\left(\sum_{k=1}^{\infty} \frac{Y_k}{2^k}\right)$$

We have that

$$\begin{aligned} \mathbb{P}\left(a \leq Z \leq b\right) &= \mathbb{P}\left(N(a) \leq N(Z) \leq N(b)\right) \\ &= \mathbb{P}\left(N(a) \leq X \leq N(b)\right) \\ &= N(b) - N(a) \end{aligned}$$

Last step follows from Exercise 1.2.5. Thus Z is normally distributed over $(\Omega_\infty, \mathcal{F}_\infty, \mathbb{P}_\infty)$. Denote

$$Z_n = N^{-1}\left(\sum_{k=1}^{n} \frac{Y_k}{2^k}\right)$$

Clearly, $\lim Z_n(\omega) = Z(\omega)$ for each $\omega \in \Omega_\infty$.