Exercise 1.5

Let $X \geq 0$ be a random variable with cumulative distribution function $F(x)$. Show that

$$\mathbb{E}X = \int_0^{+\infty} (1 - F(x))\,\mathrm{d}x.$$

Here $F(x) = \mathbb{P}(X \leq x)$.

Proof

We have that

$$\int_\Omega \int_0^{+\infty} \mathbf{1}_{[0,X(\omega))]}(x)\mathrm{d}x\mathrm{d}\mathbb{P}(\omega) = \int_0^{+\infty} \int_\Omega \mathbf{1}_{[0,X(\omega))]}(x)\mathrm{d}\mathbb{P}(\omega)\mathrm{d}x$$

RHS is

$$\begin{aligned}
\int_0^{+\infty} \int_\Omega \mathbf{1}_{[0,X(\omega))]}(x)\mathrm{d}\mathbb{P}(\omega)\mathrm{d}x &= \int_0^{+\infty} \int_\Omega \mathbf{1}_{\{x \leq X(\omega)\}}\mathrm{d}\mathbb{P}(\omega)\mathrm{d}x \\
&= \int_0^{+\infty} \mathbb{P}(x \leq X)\mathrm{d}x \\
&= \int_0^{+\infty} (1 - F(x))\,\mathrm{d}x
\end{aligned}$$

LHS is

$$\int_\Omega \int_0^{+\infty} \mathbf{1}_{[0,X(\omega))]}(x)\mathrm{d}x\mathrm{d}\mathbb{P}(\omega) = \int_\Omega X(\omega)\mathrm{d}\mathbb{P}(\omega) = \mathbb{E}X.$$

The proof is complete.