Exercise 1.6

Find the moment generating function of a normally distributed random variable. For $\varphi(x) = e^{ux}$, verify Jensen's inequality.

Proof

Let $X \sim \mathcal{N}(\mu, \sigma^2)$. We have that

$$\begin{aligned}\mathbb{E}\varphi(X) &= \mathbb{E}e^{uX} \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{ux - \frac{(x-\mu)^2}{2\sigma^2}}\end{aligned}$$

Next,

$$\begin{aligned}ux - \frac{(x-\mu)^2}{2\sigma^2} &= -\frac{1}{2\sigma^2}\left[(x-\mu)^2 - 2\sigma^2 ux\right] \\ &= -\frac{1}{2\sigma^2}\left[x^2 - 2(\mu+\sigma^2 u)x + (\mu+\sigma^2 u)^2 + \mu^2 - (\mu+\sigma^2 u)^2\right] \\ &= -\frac{1}{2\sigma^2}\left[x^2 - 2(\mu+\sigma^2 u)x + (\mu+\sigma^2 u)^2 - \sigma^2 u(2\mu+\sigma^2 u)\right] \\ &= -\frac{1}{2\sigma^2}\left[(x-\mu-\sigma^2 u)^2\right] + u\left(\mu + \frac{1}{2}\sigma^2 u\right)\end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{E}\varphi(X) &= e^{u\left(\mu+\frac{1}{2}\sigma^2 u\right)} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{\frac{(x-\mu-\sigma^2\mu)^2}{2\sigma^2}} \\ &= e^{u\mu+\frac{1}{2}\sigma^2 u^2}\end{aligned}$$

Here we used the fact that $\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} f_{\mu+\sigma^2\mu,\sigma^2}(x)\mathrm{d}x = 1$ where $f_{\mu+\sigma^2\mu,\sigma^2}$ is a density function for $Z \sim \mathcal{N}(\mu+\sigma^2\mu, \sigma^2)$. Thus

$$\begin{aligned}\mathbb{E}\varphi(X) &= e^{u\mu+\frac{1}{2}\sigma^2 u^2} \\ &\geq e^{u\mu} \\ &= \varphi(\mathbb{E}(X))\end{aligned}$$