Exercise 1.7

Normal density with mean 0 and variance n is denoted by

$$f_n(x) = \frac{1}{\sqrt{2n\pi}} e^{-\frac{x^2}{2n}}$$

1. Find $f(x) = \lim_{n\to+\infty} f_n(x)$
2. Find $\lim_{n\to+\infty} \int_{-\infty}^{+\infty} f_n(x)\mathrm{d}x$
3. Explain why MCT is not violated even if $\lim_{n\to+\infty} \int_{-\infty}^{+\infty} f_n(x)\mathrm{d}x = \int_{-\infty}^{+\infty} f(x)\mathrm{d}x$

Proof

- $\forall x$, $\lim_{n\to+\infty} f_n(x) = 0$.
- $\lim_{n\to+\infty} \int_{-\infty}^{+\infty} f_n(x)\mathrm{d}x = 1$
- For each real x, there exists positive integer $\kappa(x)$ such that for $k \geq \kappa(x)$ the following bound doesn't hold

$$f_k(x) \leq f_{k+1}(x) \iff \sqrt{1 + \frac{1}{k}} \leq e^{\frac{x^2}{2k(k+1)}}$$