Exercise 1.8 (Moment-generating function)

Consider a random variable X satisfying $\mathbb{E}[Xe^{tX}], \mathbb{E}[|X|e^{tX}] < +\infty$ for every $t \in \mathbb{R}$. Denote the moment generating function by

$$\varphi(t) = \mathbb{E}\ e^{tX}.$$

Show that $\varphi'(t) = \mathbb{E}\ Xe^{tX}$.

Proof

It suffices to consider the case where $X \geq 0$. To see this, write $X = X^+ - X^-$. Then $\mathbb{E}X^+e^{tX^+} < +\infty, \mathbb{E}X^-e^{tX^-} \leq \mathbb{E}[|X|e^{tX}] < +\infty$. We have that

$$e^{tX} = e^{tX^+} + e^{-tX^-} - 1$$

Therefore, if $\varphi_+(t) := \mathbb{E}\ e^{tX^+}$ and $\varphi_-(t) := \mathbb{E}\ e^{tX^-}$, then $\varphi(t) = \varphi_+(t) + \varphi_-(t) - 1$. Therefore, $\varphi'(t)$ must exist and it must hold that

$$\varphi'(t) = \varphi'_+(t) - \varphi'_-(-t) = \mathbb{E}\ X^+e^{tX^+} - \mathbb{E}\ X^-e^{-tX^-} = \mathbb{E}\ X^+e^{tX^+} - X^-e^{-tX^-}.$$

It is easy to verify that $Xe^{tX} = X^+e^{tX^+} - X^-e^{-tX^-}$. Thus $\varphi'(t) = \mathbb{E}Xe^{tX}$. Now suppose that $X \geq 0$. Fix t and let $s_n \to t$. Denote

$$Y_n = \frac{e^{tX} - e^{s_nX}}{t - s_n}$$

By elementary calculus, there exists a random variable $t_n(\omega)$ which lies between t and s_n and

$$Y_n(\omega) = X(\omega)e^{t_n(\omega)X(\omega)}$$

For large enough n, it must hold that $Y_n \leq Xe^{2tX}$ and since $\mathbb{E}Xe^{2tX} < +\infty$, DCT implies

$$\lim \mathbb{E}Y_n = \mathbb{E} \lim Y_n = \mathbb{E}Xe^{tX}$$

However,

$$\mathbb{E}Y_n = \frac{\varphi(t) - \varphi(s_n)}{t - s_n}$$

Thus, $\lim \mathbb{E}Y_n = \varphi'(t)$. The proof is complete.