Exercise 1.9

We say that random variable X which is defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is independent of event $A \in \mathcal{F}$ if the following holds

$$\int_A \mathbf{1}_B(X(\omega))\mathrm{d}\mathbb{P}(\omega) = \mathbb{P}(A) \cdot \mathbb{P}(X \in B) \text{ for every Borel subset } B \text{ of } \mathbb{R}.$$

Show that for any non-negative, Borel-measurable function g the following is true if X is independent of event A.

$$\int_A g(X(\omega))\mathrm{d}\mathbb{P}(\omega) = \mathbb{P}(A) \cdot \mathbb{E}g(X).$$

Proof

We know from measure theory that for some $a_i \geq 0$ and $B_i \in \mathcal{B}$ the following representation holds

$$g = \sum_{i \geq 0} a_i \mathbf{1}_{B_i}$$

We have that

$$\begin{aligned}
\int_A g(X(\omega))\mathrm{d}\mathbb{P}(\omega) &= \sum_{i\geq 0} a_i \int_A \mathbf{1}_{B_i}(X(\omega))\mathrm{d}\mathbb{P}(\omega) \\
&= \mathbb{P}(A) \cdot \sum_{i\geq 0} a_i \mathbb{P}(X \in B_i) \\
&= \mathbb{P}(A) \cdot \sum_{i\geq 0} a_i \mathbb{E}\mathbf{1}_{B_i}(X) \\
&= \mathbb{P}(A) \cdot \mathbb{E} \sum_{i\geq 0} a_i \mathbf{1}_{B_i}(X) \\
&= \mathbb{P}(A) \cdot \mathbb{E}g(X).
\end{aligned}$$