Exercise 2.10

Consider two random variables X and Y with *joint density function* $f_{X,Y}(x,y)$. Therefore, for every Borel subset C of $\mathbb{R}^2$,

$$\mathbb{P}((X,Y) \in C) = \int_C f_{X,Y}(x,y)\mathrm{d}x\mathrm{d}y.$$

In then holds that

$$\mathbb{E}[Y|X=x] = \int_{-\infty}^{+\infty} y f_{Y|X}(y|x)\mathrm{d}y$$

Here $f_{Y|X}(y|x)$ denotes the *conditional density* defined by

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

where $f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,\eta)\mathrm{d}\eta$ is the *marginal density* of X. Define

$$g(x) = \int_{-\infty}^{+\infty} y f_{Y|X}(y|x)\mathrm{d}y.$$

Show that if $\mathbb{E}|Y| < +\infty$, then $\mathbb{E}[Y|X] = g(X)$.

Proof

For a Borel subset B of $\mathbb{R}$, let $A = X^{-1}(B)$. We need to show that

$$\int_A g(X)\mathrm{d}\mathbb{P} = \int_A Y\mathrm{d}\mathbb{P}.$$

We have that

$$\begin{aligned}
\int_A g(X(\omega))\mathrm{d}\mathbb{P}(\omega) &= \int_{-\infty}^{+\infty} \mathbf{1}_B(x)g(x)f_X(x)\mathrm{d}x \\
&= \int_{-\infty}^{+\infty} \mathbf{1}_B(x)f_X(x)\left(\int_{-\infty}^{+\infty} \frac{yf_{X,Y}(x,y)}{f_X(x)}\mathrm{d}y\right)\mathrm{d}x \\
&\overset{(1)}{=} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathbf{1}_B(x)f_X(x) \cdot \frac{yf_{X,Y}(x,y)}{f_X(x)}\mathrm{d}x\mathrm{d}y \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y\mathbf{1}_B(x)f_{X,Y}(x,y)\mathrm{d}x\mathrm{d}y \\
&= \mathbb{E}[\mathbf{1}_B(X)Y] \\
&= \int_\Omega \mathbf{1}_B\left(X(\omega)\right)Y(\omega)\mathrm{d}\mathbb{P}(\omega) \\
&\overset{(2)}{=} \int_A \mathbf{1}_A(\omega)Y(\omega)\mathrm{d}\mathbb{P}(\omega) \\
&= \int_A Y(\omega)\mathrm{d}\mathbb{P}(\omega)
\end{aligned}$$

Here in (1), we used Fubini's theorem. Note that

$$\begin{aligned}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left|\mathbf{1}_B(x)\cdot y f_{X,Y}(x,y)\right| \mathrm{d}x\mathrm{d}y &\le \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |y|\, f_{X,Y}(x,y)\mathrm{d}x\mathrm{d}y \\ &= \mathbb{E}|Y| < +\infty.\end{aligned}$$

Also in (2) we used the following

$$\mathbf{1}_B\left(X(\omega)\right)Y(\omega) = \begin{cases} Y(\omega) & \text{if } \omega \in A \\ 0 & \text{otherwise} \end{cases}$$