Exercise 2.11

Let W be $\sigma(X)$-measurable and non-negative. Show that $W = g(X)$ for some function g. In particular, if Y is a non-negative random variable, then there exists some function g such that $\mathbb{E}[Y|X] = g(X)$.

Proof

We know from measure theory that for some $a_i \geq 0$ and $A_i \in \sigma(X)$ the following representation holds

$$W = \sum_{i \geq 0} a_i \mathbf{1}_{A_i}$$

Suppose that $A_i = X^{-1}(B_i)$ for some Borel-measurable subset B_i. Denote by

$$g := \sum_{i \geq 0} a_i \mathbf{1}_{B_i}.$$

Since $\mathbf{1}_{B_i}(X(\omega)) = \mathbf{1}_{A_i}(\omega)$, it then holds that

$$g(X(\omega)) = \sum_{i \geq 0} a_i \mathbf{1}_{B_i}(X(\omega)) = \sum_{i \geq 0} a_i \mathbf{1}_{A_i}(\omega) = W(\omega).$$

To see the second part, note that $\mathbb{E}[Y|X]$ is $\sigma(X)$-measurable and non-negative.