## Exercise 2.2

Let $\Omega = \{HH, HT, TH, TT\}$ and $S_0 = 4, S_1(H) = 8, S_1(T) = 2$. Consider two probability measures

$$\mathbb{P}(HH) = \frac{4}{9} \quad \mathbb{P}(HT) = \frac{2}{9} \qquad\qquad \mathbb{P}(TH) = \frac{2}{9} \quad \mathbb{P}(TT) = \frac{1}{9}$$
$$\tilde{\mathbb{P}}(HH) = \frac{1}{4} \quad \tilde{\mathbb{P}}(HT) = \frac{1}{4} \qquad\qquad \tilde{\mathbb{P}}(TH) = \frac{1}{4} \quad \tilde{\mathbb{P}}(TT) = \frac{1}{4}$$

Finally, define

$$X = \begin{cases} 1 & S_2 = 4 \\ 0 & \text{otherwise} \end{cases}$$

Find $\sigma(X)$ and $\sigma(S_1)$ and discuss whether they are dependent or independent under $\mathbb{P}$ and $\tilde{\mathbb{P}}$.

## Proof

Note that

$$\sigma(X) = \{\emptyset, \Omega, \underbrace{\{HT, TH\}}_{X=1}, \underbrace{\{HH, TT\}}_{X=0}\}$$
$$\sigma(S_1) = \{\emptyset, \Omega, \underbrace{\{HH, HT\}}_{S_1=H}, \underbrace{\{TH, TT\}}_{S_1=T}\}$$

Then

$$\begin{aligned} \frac{2}{9} = \mathbb{P}(TH) &= \mathbb{P}(\{S_1 = T\} \cap \{X = 1\}) \\ &\neq \mathbb{P}(\{S_1 = T\})\, \mathbb{P}(\{X = 1\}) \\ &= \mathbb{P}(TH, TT)\, \mathbb{P}(\{HT, TH\}) \\ &= \frac{3}{9} \cdot \frac{4}{9} \\ &= \frac{4}{27} \end{aligned}$$

In order to show $\sigma(X)$ and $\sigma(S_1)$ are independent, it suffices to show that

$$\tilde{\mathbb{P}}(\{X = 1\} \cap \{S_1 = H\}) = \tilde{\mathbb{P}}(\{X = 1\})\, \tilde{\mathbb{P}}(\{S_1 = H\})$$

This is because when $A$ and $B$ are independent, then

$$\begin{aligned} \mathbb{P}(A^c \cap B) &= \mathbb{P}(B) - \mathbb{P}(A \cap B) \\ &= \mathbb{P}(B) - \mathbb{P}(A)\mathbb{P}(B) \\ &= \mathbb{P}(B)(1 - \mathbb{P}(A)) \\ &= \mathbb{P}(A^c)\mathbb{P}(B) \end{aligned}$$

We finally have that

$$\begin{aligned} \frac{1}{4} = \tilde{\mathbb{P}}(HT) &= \tilde{\mathbb{P}}(\{X = 1\} \cap \{S_1 = H\}) \\ &= \frac{1}{2} \cdot \frac{1}{2} \\ &= \tilde{\mathbb{P}}(\{X = 1\})\, \tilde{\mathbb{P}}(\{S_1 = H\}) \end{aligned}$$