Exercise 2.3 (Rotating the axes)

Let X and Y be independent standard normal random variables. For constant θ, define

$$V = X\cos\theta + Y\sin\theta \text{ and } W = -X\sin\theta + Y\cos\theta.$$

Show that V and W are independent standard normal random variables.

Proof

We have that

$$\begin{aligned}
\mathbb{E}\ e^{\alpha V + \beta W} &= \mathbb{E}\ e^{(\alpha\cos\theta - \beta\sin\theta)X + (\alpha\sin\theta + \beta\cos\theta)Y} \\
&= \mathbb{E}\ e^{(\alpha\cos\theta - \beta\sin\theta)X} \cdot \mathbb{E}\ e^{(\alpha\sin\theta + \beta\cos\theta)Y} \\
&= e^{\frac{1}{2}(\alpha\cos\theta - \beta\sin\theta)^2} \cdot e^{\frac{1}{2}(\alpha\sin\theta + \beta\cos\theta)^2} \\
&= e^{\frac{1}{2}(\alpha^2 + \beta^2)}
\end{aligned}$$

Set $\alpha = 0$ and $\beta = 0$ to show W and V are standard normal random variables resp. We then see

$$\mathbb{E}\ e^{\alpha V + \beta W} = \mathbb{E}\ e^{\alpha V} \cdot \mathbb{E}\ e^{\beta W}$$

Therefore V and W are independent as well.