Exercise 2.4

Suppose that $X \sim N(0,1)$ and Z is a random variable with Rademacher distribution *i.e.,*

$$\mathbb{P}(Z=-1)=\mathbb{P}(Z=1)=\tfrac{1}{2}$$

Show that $Y = XZ \sim N(0,1)$, but X and Y are not independent.

Proof

We have that

$$\begin{aligned}
\mathbb{E}e^{uX+vY} &= \mathbb{E}e^{uX+vXZ} \\
&= \mathbb{E}e^{(u+vZ)X} \\
&= \mathbb{E}[\mathbb{E}e^{(u+vZ)X}|X] \\
&= \tfrac{1}{2}\mathbb{E}e^{X(u+v)} + \tfrac{1}{2}\mathbb{E}e^{X(u-v)} \\
&= \tfrac{1}{2}e^{\frac{(u+v)^2}{2}} + \tfrac{1}{2}e^{\frac{(u-v)^2}{2}} \\
&= e^{\frac{u^2+v^2}{2}} \cdot \frac{e^{uv}+e^{-uv}}{2}
\end{aligned}$$

Set $u = 0$ to get

$$\mathbb{E}e^{vY} = e^{\frac{v^2}{2}}.$$

Thus, $Y \sim N(0,1)$. However, note that

$$\mathbb{E}e^{uX+vY} \neq \mathbb{E}e^{uX} \cdot \mathbb{E}e^{vY}$$

Hence X and Y are not independent.