Exercise 2.5

Proof

First note that

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\mathrm{d}y = \int_{-\infty}^{+\infty} f_{X,Y}(-x,y)\mathrm{d}y = f_X(-x).$$

Suppose that $x \geq 0$. We have that

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\mathrm{d}y \\ &= \int_{-x}^{+\infty} \tfrac{2x+y}{\sqrt{2\pi}} \cdot e^{-\frac{(2x+y)^2}{2}}\mathrm{d}y \\ &= \int_{x}^{+\infty} \tfrac{u}{\sqrt{2\pi}} \cdot e^{-\frac{u^2}{2}}\mathrm{d}u \\ &= \tfrac{-1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} |_x^{+\infty} \\ &= \tfrac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \end{aligned}$$

Similarly, since $f_{X,Y}(x,y) = f_{X,Y}(-x,y)$, we have that

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\mathrm{d}x \\ &= 2\int_{0}^{+\infty} f_{X,Y}(x,y)\mathrm{d}x \\ &= 2\int_{\ell}^{+\infty} \tfrac{2x+y}{\sqrt{2\pi}} \cdot e^{-\frac{(2x+y)^2}{2}}\mathrm{d}x \qquad \text{here } \ell := \max\{0,-y\} \\ &= \int_{2\ell+y}^{+\infty} \tfrac{u}{\sqrt{2\pi}} \cdot e^{-\frac{u^2}{2}}\mathrm{d}x \\ &= \tfrac{-1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} |_{2\ell+y}^{+\infty} \\ &= \tfrac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \end{aligned}$$

Here we use that $2\ell + y = |y|$. To see that X and Y are not independent, note that

$$f_X(x) \times f_Y(y) \neq f_{X,Y}(x,y).$$

This is true because RHS is zero for some pairs of (x,y), but LHS is never zero. It remains to show that X and Y are uncorrelated *i.e*, $\mathbb{E}XY = \mathbb{E}X\mathbb{E}Y$. Letting $u = -x$, we get

$$\begin{aligned} \mathbb{E}XY &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xyf_{X,Y}(x,y)\mathrm{d}x\mathrm{d}y \\ &= -\int_{-\infty}^{+\infty}\int_{+\infty}^{-\infty} (-u)yf_{X,Y}(-u,y)\mathrm{d}u\mathrm{d}y \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (-u)yf_{X,Y}(u,y)\mathrm{d}u\mathrm{d}y = -\mathbb{E}XY. \end{aligned}$$

Thus $\mathbb{E}XY = 0$. The proof is complete.