Exercise 2.6

Let $\Omega = \{a, b, c, d\}$. Define

$$\mathbb{P}(a) = \frac{1}{6}, \quad \mathbb{P}(b) = \frac{1}{3}, \quad \mathbb{P}(c) = \frac{1}{4}, \quad \mathbb{P}(d) = \frac{1}{4}$$

Moreover,

$$X(a) = 1, \quad X(b) = 1, \quad X(c) = -1, \quad X(d) = -1,$$
$$Y(a) = 1, \quad Y(b) = -1, \quad Y(c) = 1, \quad Y(d) = -1$$

Set $Z = X + Y$. Write down $\sigma(X)$ and compute $\mathbb{E}[Z|X], \mathbb{E}[Y|X]$. Verify that $\mathbb{E}[Z|X] - \mathbb{E}[Y|X] = X$.

Proof

Besides Ω and $\emptyset$, the following two atoms belong to $\sigma(X)$

$$\{X = 1\} = \{a, b\}, \quad \{X = -1\} = \{c, d\}.$$

We have that

$$\begin{aligned}\mathbb{E}[Z|X = 1] &= \sum z f_{Z|X}(z|1)\mathrm{d}z \\ &= 2 \cdot \frac{\frac{1}{6}}{\frac{1}{2}} + 0 \cdot \frac{\frac{1}{3}}{\frac{1}{2}} \\ &= \frac{2}{3}\end{aligned}$$

Also

$$\begin{aligned}\mathbb{E}[Z|X = -1] &= \sum z f_{Z|X}(z|-1)\mathrm{d}z \\ &= 0 \cdot \frac{\frac{1}{4}}{\frac{1}{2}} - 2 \cdot \frac{\frac{1}{4}}{\frac{1}{2}} \\ &= -1\end{aligned}$$

Next,

$$\begin{aligned}\mathbb{E}[Y|X = 1] &= \sum y f_{Y|X}(y|1)\mathrm{d}y \\ &= 1 \cdot \frac{\frac{1}{6}}{\frac{1}{2}} - 1 \cdot \frac{\frac{1}{3}}{\frac{1}{2}} \\ &= \frac{1}{3} - \frac{2}{3} \\ &= -\frac{1}{3}\end{aligned}$$

Finally,

$$\begin{aligned}\mathbb{E}[Y|X = -1] &= \sum y f_{Y|X}(y|-1)\mathrm{d}y \\ &= 1 \cdot \frac{\frac{1}{4}}{\frac{1}{2}} - 1 \cdot \frac{\frac{1}{4}}{\frac{1}{2}} \\ &= 0\end{aligned}$$

Therefore,

$$\mathbb{E}[Z|X=1] - \mathbb{E}[Y|X=1] = \frac{2}{3} + \frac{1}{3} = 1 = \mathbb{E}[X|X=1]$$
$$\mathbb{E}[Z|X=-1] - \mathbb{E}[Y|X=-1] = -1 - 0 = -1 = \mathbb{E}[X|X=-1]$$

Putting pieces together,

$$\mathbb{E}[Z|X=x] - \mathbb{E}[Y|X=x] = x.$$

This is expected as $\mathbb{E}[Z|X=x] - \mathbb{E}[Y|X=x] = \mathbb{E}[X|X=x] = x$ or $\mathbb{E}[Z|X] - \mathbb{E}[Y|X] = X$.