Exercise 2.7

Let Y to be an integrable random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$ and let X be an arbitrary $\mathcal{G}$-measurable random variable. Show that

$$\operatorname{Var}(Y - \mathbb{E}[Y|\mathcal{G}]) \leq \operatorname{Var}(Y - X)$$

Proof

Define $\mu := \mathbb{E}(Y - X)$. We have that

$$\begin{aligned}
\operatorname{Var}(Y - X) &= \mathbb{E}(Y - X - \mu)^2 \\
&= \mathbb{E}\left((Y - \mathbb{E}[Y|\mathcal{G}]) + (\mathbb{E}[Y|\mathcal{G}] - X - \mu)\right)^2 \\
&= \mathbb{E}(Y - \mathbb{E}[Y|\mathcal{G}])^2 + \mathbb{E}(\mathbb{E}[Y|\mathcal{G}] - X - \mu)^2 + 2\mathbb{E}\left[\underbrace{(\mathbb{E}[Y|\mathcal{G}] - X - \mu)}_{:=Z_1}\underbrace{(Y - \mathbb{E}[Y|\mathcal{G}])}_{:=Z_2}\right]
\end{aligned}$$

Since Z_1 is $\mathcal{G}$-measurable, it holds that

$$\begin{aligned}
\mathbb{E}[Z_1 Z_2|\mathcal{G}] &= Z_1 \mathbb{E}[Z_2|\mathcal{G}] \\
&= Z_1 (\mathbb{E}[Y|\mathcal{G}] - \mathbb{E}[Y|\mathcal{G}]) \\
&= 0
\end{aligned}$$

Thus, $\mathbb{E}[Z_1 Z_2] = \mathbb{E}[\mathbb{E}[Z_1 Z_2|\mathcal{G}]] = 0$. Continuing,

$$\begin{aligned}
\operatorname{Var}(Y - X) &= \mathbb{E}(Y - \mathbb{E}[Y|\mathcal{G}])^2 + \mathbb{E}(\mathbb{E}[Y|\mathcal{G}] - X - \mu)^2 \\
&\geq \mathbb{E}(Y - \mathbb{E}[Y|\mathcal{G}])^2 \\
&= \operatorname{Var}(Y - \mathbb{E}[Y|\mathcal{G}]).
\end{aligned}$$

The proof is complete.