Exercise 2.8

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let X and Y be integrable random variables on this space. Denote by $Z = Y - \mathbb{E}[Y|X]$. Show that Z is uncorrelated with any $\sigma(X)$-measurable random variable.

Proof

Let W be an arbitrary $\sigma(X)$-measurable. We have that

$$\mathbb{E}[ZW] = \mathbb{E}[\mathbb{E}[ZW|X]] = \mathbb{E}[W\mathbb{E}[Z|X]]$$

On the other hand,

$$\mathbb{E}[Z|X] = \mathbb{E}[Y|X] - \mathbb{E}[Y|X] = 0 \text{ and } \mathbb{E}[Z] = \mathbb{E}[\mathbb{E}[Z|X]] = \mathbb{E}[0] = 0.$$

Thus,

$$\mathbb{E}[ZW] = 0 = \underbrace{\mathbb{E}[Z]}_{=0}\,\mathbb{E}[W]$$