Exercise 2.9

Give an example of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a random variable X defined on this space and a function f such that

$$\sigma(f(X)) \subsetneq \sigma(X).$$

Is it possible to have $\sigma(f(X)) \not\subseteq \sigma(X)$?

Proof

Set

$$\Omega = \mathbb{R}, \quad \mathcal{F} = \mathcal{B}, \quad \mathbb{P}(A) = \frac{1}{\sqrt{2\pi}} \int_A e^{-\frac{t^2}{2}} \mathrm{d}t.$$

Define $X(\omega) = \omega$ and $f(\omega) = \omega^2$. Then we have that

$$\sigma(X) = \{\{\omega \in \Omega : X(\omega) \in C\} : C \in \mathcal{B}\} = \mathcal{B}$$

Furthermore,

$$\begin{aligned}
\sigma(f(X)) &= \{\{\omega \in \Omega : f(X)(\omega) \in C\} : C \in \mathcal{B}\} \\
&= \left\{\{\omega \in \Omega : \omega^2 \in C\} : C \in \mathcal{B}\right\} \\
&= \left\{\{\omega \in \Omega : \omega^2 \in C\} : C \in \mathcal{B} \text{ and } C \cap \mathbb{R}^{\geq 0} \neq \emptyset\right\}
\end{aligned}$$

Note that it always holds that $\sigma(f(X)) \subseteq \sigma(X)$. To see this, note that

$$\begin{aligned}
\sigma(f(X)) &= \{\{\omega \in \Omega : f(X)(\omega) \in C\} : C \in \mathcal{B}\} \\
&= \left\{\{\omega \in \Omega : X(\omega) \in f^{-1}(C)\} : C \in \mathcal{B}\right\} \\
&\subseteq \left\{\{\omega \in \Omega : X(\omega) \in f^{-1}(C)\} : f^{-1}(C) \in \mathcal{B}\right\} \\
&\subseteq \{\{\omega \in \Omega : X(\omega) \in C\} : C \in \mathcal{B}\} = \sigma(X).
\end{aligned}$$

Here we used the fact that $f^{-1}(C) \in \mathcal{B}$ whenever $B \in \mathcal{B}$.