Exercise 3.1

Adopt notation from Definition 3.3.3. Show that, for $0 \le t < u_1 < u_2$, the increment $W(u_2) - W(u_1)$ is also independent of $\mathcal{F}(t)$.

Proof

Recall that property (i) of Definition 3.3.3 states the following:

For $0 \le s < t$, every set in $\mathcal{F}(s)$ is also in $\mathcal{F}(t)$.

We now notice that $W(u_2) - W(u_1)$ is independent of $\mathcal{F}(u_1)$ because of property (iii). Now since every set in $\mathcal{F}(t)$ lies inside $\mathcal{F}(u_1)$ as well, due to property (i), we conclude that $W(u_2) - W(u_1)$ is also independent of $\mathcal{F}(t)$.

Remark: We **cannot** conclude the result without using property (i). It is indeed tempting to argue that since

$$W(u_2) - W(u_1) = \underbrace{W(u_2) - W(t)}_{:=X} - (\underbrace{W(u_1) - W(t)}_{:=Y}).$$

Next, by property (iii), we know that both random variables X and Y are independent of $\mathcal{F}(t)$. Then from here one may try to conclude that $X - Y$ is therefore independent of $\mathcal{F}(t)$ as well. However, this conclusion will be **incorrect**. Example below may help to see why!

Example: Note that if X and Y are independent of Z, it is not true in general that $X + Y$ is also independent of Z. An an example, let X and Y be outcome of tossing a fair coin and define $Z = XY$. It is easy to show that X and Y both are independent of Z, but $X + Y$ is not. In fact,

$$\begin{aligned}
\mathbb{P}(Z = 1, X + Y = 0) &= \mathbb{P}(X^2 = -1, X = -Y) \\
&= 0 \neq \frac{1}{4} \\
&= \mathbb{P}(Z = 1)\mathbb{P}(X + Y = 0).
\end{aligned}$$