Exercise 3.2

Let $W(t)$, $t \geq 0$, be a Brownian motion, and let $\mathcal{F}(t), t \geq 0$, be a filtration for this Brownian motion. Show that $W^2(t) - t$ is a martingale. (Hint: For $0 \leq s < t$, write $W^2(t)$ as $(W(t) - W(s))^2 + 2W(t)W(s) - W^2(s)$.)

Proof

Following the hint, for $0 \leq s < t$, we write

$$\begin{aligned}
\mathbb{E}[W^2(t)|\mathcal{F}(s)] &= \mathbb{E}[(W(t) - W(s))^2 + 2W(t)W(s) - W^2(s)|\mathcal{F}(s)] \\
&= \mathbb{E}[(W(t) - W(s))^2] + 2W(s) \cdot \mathbb{E}[W(t)|\mathcal{F}(s)] - W^2(s) \\
&= \mathbb{E}[(W(t) - W(s))^2] + 2W(s)W(s) - W^2(s) \\
&= \mathbb{E}[(W(t) - W(s))^2] + W^2(s) \\
&= \mathrm{Var}(W(t) - W(s)) + W^2(s) \\
&= t - s + W^2(s)
\end{aligned}$$

Rearranging, we obtain that

$$\mathbb{E}[W^2(t) - t|\mathcal{F}(s)] = W^2(s) - s.$$