Exercise 3.3

The kurtosis of a random variable is defined to be the ratio of its fourth central moment to the square of its variance. For a normal random variable, the kurtosis is 3. Verifies this fact.

Proof

Let X be a normal random variable with mean μ so that $X-\mu$ has mean zero. Let the variance of X, which is also the variance of $X-\mu$, be σ^2. We know that

$$\phi(u) := \mathbb{E}[e^{u(X-\mu)}] = e^{\frac{u^2\sigma^2}{2}} \quad \forall u \in R.$$

Thus,

$$\begin{aligned}\phi'(u) &= \mathbb{E}\left[(X-\mu)e^{u(X-u)}\right] \\ &= \sigma^2 u \cdot e^{\frac{u^2\sigma^2}{2}}\end{aligned}$$

Continuing,

$$\begin{aligned}\phi''(u) &= \frac{\partial}{\partial u}\mathbb{E}\left[(X-\mu)e^{u(X-u)}\right] \\ &= \mathbb{E}\left[(X-\mu)^2 e^{u(X-u)}\right] \\ &= \sigma^2(1+\sigma^2u^2)\cdot e^{\frac{u^2\sigma^2}{2}}\end{aligned}$$

Next,

$$\begin{aligned}\phi'''(u) &= \frac{\partial}{\partial u}\mathbb{E}\left[(X-\mu)^2 e^{u(X-u)}\right] \\ &= \mathbb{E}\left[(X-\mu)^3 e^{u(X-u)}\right] \\ &= \sigma^2\cdot\left[\sigma^2u(1+\sigma^2u^2)+2\sigma^2u\right]\cdot e^{\frac{u^2\sigma^2}{2}}\end{aligned}$$

Finally,

$$\begin{aligned}\phi''''(u) &= \frac{\partial}{\partial u}\mathbb{E}\left[(X-\mu)^3 e^{u(X-u)}\right] \\ &= \mathbb{E}\left[(X-\mu)^4 e^{u(X-u)}\right] \\ &= \sigma^4 u\cdot\left[\sigma^2u(1+\sigma^2u^2)+2\sigma^2u\right]\cdot e^{\frac{u^2\sigma^2}{2}} + \sigma^2\cdot\left[\sigma^2+3u^2\sigma^4+2\sigma^2\right]\cdot e^{\frac{u^2\sigma^2}{2}}\end{aligned}$$

In conclusion, the following is true

$$\phi'(0)=0,\ \phi''(0)=\sigma^2,\ \phi'''(0)=0\ \phi''''(0)=3\sigma^4.$$

At the end, we have that

$$\kappa := \frac{\mathbb{E}(X-\mu)^4}{\mathrm{Var}(X)^2} = \frac{\phi''''(0)}{\phi''(0)^2} = \frac{3\sigma^4}{\sigma^4} = 3.$$

Remark: It is noteworthy to mention Stein's Lemma here. Calculations above are straightforward in view of Stein's Lemma which is described below:

Stein's Lemma: Suppose X is a normally distributed random variable with expectation μ and variance σ^2. Further suppose g is a function for which the two expectations $\mathbb{E}(g(X)(X-\mu))$ and $\mathbb{E}(g'(X))$ both exist. The following identity then holds:

$$\mathbb{E}\left(g(X)\cdot(X-\mu)\right)=\sigma^2\cdot\mathbb{E}\left(g'(X)\right)$$