Exercise 3.4

Adopt notation from Theorem 3.4.3. Show that as the number n of partition points approaches infinity and the length of the longest subinterval approaches zero, the sample first variation

$$\sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|$$

approaches infinity for almost every path of the Brownian motion W. (Hint:

$$\sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2 \leq \max_{0 \leq k \leq n-1} |W(t_{k+1}) - W(t_k)| \cdot \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|)$$

(ii) Show that as the number n of partition points approaches $+\infty$ and the length of the longest sub interval approaches zero, the sample cubic variation

$$\sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^3$$

approaches zero for almost every path of the Brownian motion W.

Proof (i): Following the hint, we have that

$$\begin{aligned} \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2 &\leq \max_{0 \leq k \leq n-1} |W(t_{k+1}) - W(t_k)| \cdot \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)| \\ &\leq \|\Pi\| \cdot \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)| \end{aligned}$$

For $c > 0$, define

$$A_c := \{\omega : \lim_{n \to +\infty, \|\Pi\| \to 0} \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)| < c\}$$

Note that for $\omega \in A_c$, it holds that

$$T = \lim_{n \to +\infty} \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2 \leq 0 \cdot c = 0$$

Since $T > 0$, we conclude that $\mathbb{P}(A_c) = 0$. Hence,

$$\{\omega : \lim_{n \to +\infty, \|\Pi\| \to 0} \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)| < +\infty\} = \cup_{m=1}^{+\infty} A_m$$

has probability zero.

Proof (ii): Similar to (i), we write

$$\begin{aligned}\sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^3 &\leq \max_{0 \leq k \leq n-1} |W(t_{k+1}) - W(t_k)| \cdot \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2 \\ &\leq \|\Pi\| \cdot \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2\end{aligned}$$

Taking limits, we obtain that

$$\lim_{n\to+\infty, \|\Pi\|\to 0} \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^3 \leq \lim_{n\to+\infty, \|\Pi\|\to 0} \|\Pi\| \cdot \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2 = 0$$

The right hand side equality holds since $T = \lim_{n\to+\infty, \|\Pi\|\to 0} \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2 < +\infty$ *a.s.* To be more rigorous, there exists $\epsilon > 0$ such that for $\|\Pi\| < \epsilon$, it will hold almost surely that $\sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2 < 2T$. Hence,

$$\lim_{n\to+\infty, \|\Pi\|\to 0} \|\Pi\| \cdot \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2 \leq \lim_{\|\Pi\|\to 0} 2T \cdot \|\Pi\| = 0.$$