Exercise 3.5

Let

$$S(t) = S(0) \cdot \exp\left((r - \frac{1}{2}\sigma^2)t + \sigma W(t)\right)$$

Here r is the interest rate, σ is the volatility. Let K be the strike price and T time to maturity. Then show that

$$\mathbb{E}\left[e^{-rT} \cdot (S(T) - K)^+\right] = S(0) \cdot N(d_+(T, S(0)) - Ke^{-rT} \cdot N(d_-(T, S(0))$$

where

$$d_\pm(T, S(0)) = \frac{1}{\sigma\sqrt{T}} \cdot \left[\log \frac{S(0)}{K} + \left(r \pm \frac{\sigma^2}{2}\right) T\right].$$

Remark: Note that this exercise computes the call option price in the risk-neutral world (*i.e.,* $\mu = r$). The real world price will be the same, however this exercise does not show that.

Proof: Reparametrize as $K \leftarrow e^{K_0} S(0)$. Rewriting and cancelling out $S(0)$ from both sides, we need to show that

$$\begin{aligned}\mathbb{E}\left[\left(\exp\left((r - \frac{1}{2}\sigma^2)T + \sigma W(T)\right) - e^{K_0}\right)^+\right] &= e^{rT} \cdot N(d_+(T, S(0)) - e^{K_0} \cdot N(d_-(T, S(0)) \\ &= e^{rT} \cdot N\left(\frac{-K_0 + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - e^{K_0} \cdot N\left(\frac{-K_0 + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)\end{aligned}$$

Factoring out e^{K_0}, this is also equivalent to

$$\begin{aligned}&\mathbb{E}\left[\left(\exp\left((r - \frac{1}{2}\sigma^2)T - K_0 + \sigma W(T)\right) - 1\right)^+\right] \\ &= e^{rT - K_0} \cdot N\left(\frac{-K_0 + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - N\left(\frac{-K_0 + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)\end{aligned}$$

Denote $a = rT - K_0$, $w_1 = a + \frac{T\sigma^2}{2}$ and $w_2 = a - \frac{T\sigma^2}{2}$. We want to show that

$$\mathbb{E}\left[(\exp(w_2 + \sigma W(T)) - 1)^+\right] = e^a \cdot N\left(\frac{w_1}{\sigma\sqrt{T}}\right) - N\left(\frac{w_2}{\sigma\sqrt{T}}\right)$$

Denote by

$$p(z) := \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{z^2}{2}}.$$

Completing the square, we get that

$$\begin{aligned}\sqrt{2\pi} \cdot e^{w_2 + \sigma\sqrt{T}z} \cdot p(z) &= e^{w_2 + \sigma\sqrt{T}z - \frac{z^2}{2}} \\ &= e^{-\frac{1}{2}\cdot\left(z^2 - 2\sigma\sqrt{T}z + \sigma^2 T\right)} \cdot e^{\frac{\sigma^2 T}{2} + w_2} \\ &= e^{-\frac{1}{2}\cdot\left(z - \sigma\sqrt{T}\right)^2} \cdot e^a\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbb{E}\left[(\exp(w_2+\sigma W(T))-1)^+\right] &= \frac{e^a}{\sqrt{2\pi}}\int_{\frac{-w_2}{\sigma\sqrt{T}}}^{+\infty} e^{-\frac{1}{2}\cdot\left(z-\sigma\sqrt{T}\right)^2}dz - \int_{\frac{-w_2}{\sigma\sqrt{T}}}^{+\infty} p(z)dz \\
&= e^a N\left(\frac{w_2}{\sigma\sqrt{T}}+\sigma\sqrt{T}\right) - N\left(\frac{w_2}{\sigma\sqrt{T}}\right) \\
&= e^a N\left(\frac{w_1}{\sigma\sqrt{T}}\right) - N\left(\frac{w_2}{\sigma\sqrt{T}}\right)
\end{aligned}$$