Exercise 3.6

(i) Define

$$X(t) = \mu t + W(t).$$

Show that for any Borel-measurable function $f(y)$, and for any $0 \le s < t$, the function

$$g(x) = \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{+\infty} f(y) \cdot \exp\left(-\frac{(y - x - \mu(t-s))^2}{2(t-s)}\right) dy$$

satisfies $\mathbb{E}[f(X(t))|\mathcal{F}(s)] = g(X(s))$, and hence X has the Markov property.

(ii) Consider the geometric Brownian motion

$$S(t) = S(0)e^{\sigma W(t) + \nu t}.$$

Set $\tau = t - s$ and

$$p(\tau, x, y) = p(\tau, x, y) = \frac{1}{\sigma y \sqrt{2\pi\tau}} \exp\left(-\frac{\left(\log \frac{y}{x} - \nu\tau\right)^2}{2\sigma^2\tau}\right).$$

Show that $\mathbb{E}[f(S(t))|\mathcal{F}(s)] = g(S(s))$ where $g(x) = \int_{-\infty}^{+\infty} f(y)p(\tau, x, t)dy$. Thus, S has the Markov property and $p(\tau, x, y)$ is the transition probability.

Proof

(i) The proof is similar to the zero-drift case. The key fact used is the Independence Lemma which is stated below:

Independence Lemma: Suppose that X is $\mathcal{G}$-measurable and Y is independent of $\mathcal{G}$. Then the following is true:

$$\mathbb{E}[f(X, Y)|\mathcal{G}] = g(X) \quad \text{where } g(x) = \mathbb{E}[f(x, Y)].$$

We now write

$$\mathbb{E}[f(X(t))|\mathcal{F}(s)] = \mathbb{E}[f(\underbrace{X(s)}_{:=Z_1} + \underbrace{X(t) - X(s)}_{:=Z_2})|\mathcal{F}(s)]$$

Note that Z_1 is $\mathcal{F}(s)$-measurable and Z_2 is independent of $\mathcal{F}(s)$. We have that

$$g(z_1) = \mathbb{E}[f(z_1 + Z_2)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(z_1 + w) \cdot e^{-\frac{(w - \mu(t-s))^2}{2(t-s)^2}} dw.$$

Here we used the fact that $Z_2 \sim N(\mu(t-s), (t-s)^2)$. Reparamerization gives

$$g(z_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y) \cdot e^{-\frac{(y - z_1 - \mu(t-s))^2}{2(t-s)^2}} dy.$$

Applying Independence Lemma yields that

$$\mathbb{E}[f(X(t))|\mathcal{F}(s)] = \mathbb{E}[f(Z_1 + Z_2)|\mathcal{F}(s)] = g(Z_1) = g(X(s)).$$

(ii) The result follows from Independence Lemma again. Notice that

$$\mathbb{E}[f(S(t))|\mathcal{F}(s)] = \mathbb{E}\left[f\left(\underbrace{S(0)e^{\sigma W(s)+\nu s}}_{:=Z_1} \cdot \underbrace{e^{\sigma[W(t)-W(s)]+\nu(t-s)}}_{:=Z_2}\right)|\mathcal{F}(s)\right]$$

Once again Z_1 is $\mathcal{F}(s)$-measurable and Z_2 is independent of $\mathcal{F}(s)$. Denote

$$g(S(s)) = \mathbb{E}\left[f(S(s)\cdot Z_2)\right] = \frac{1}{\sqrt{2\pi\tau}}\int_{-\infty}^{+\infty} f(S(s)\cdot e^{\sigma z+\nu\tau})\cdot e^{-\frac{z^2}{2(t-s)}}\,dz.$$

Consider the following change of variables:

$$y = S(s)\cdot e^{\sigma z+\nu\tau}.$$

Thus,

$$\ln y = \ln S(s) + \sigma z + \nu\tau \Rightarrow dz = \frac{dy}{\sigma y}.$$

Continuing, we have that

$$g(S(s)) = \frac{1}{\sigma y\sqrt{2\pi\tau}}\int_{-\infty}^{+\infty} f(y)\exp\left(-\frac{\left(\ln\frac{y}{S(s)} - \nu\tau\right)^2}{2\sigma^2\tau}\right)dy.$$

From Independence Lemma, it follows that

$$\mathbb{E}[f(X(t))|\mathcal{F}(s)] = \frac{1}{\sigma y\sqrt{2\pi\tau}}\int_{-\infty}^{+\infty} f(y)\exp\left(-\frac{\left(\ln\frac{y}{S(s)} - \nu\tau\right)^2}{2\sigma^2\tau}\right)dy.$$