Exercise 3.7

Let $W(t), t \geq 0$ be a Brownian motion. Fix $m > 0$ and $\mu \in \mathbb{R}$. For $0 \leq t < +\infty$, define

$$X(t) = \mu t + W(t), \quad \tau_m = \min\{t \geq 0 : X(t) = m\}.$$

Set $\tau_m = +\infty$ if $X(t)$ never reaches m. Let σ be a positive number and set

$$Z(t) = e^{\sigma X(t) - \left(\sigma\mu + \frac{\sigma^2}{2}\right)t}.$$

(i) Show that $Z(t), t \geq 0$, is a martingale.

(ii) Use (i) to conclude that

$$\mathbb{E} \exp\left(\sigma X(t \wedge \tau_m) - \left(\sigma\mu + \frac{\sigma^2}{2}\right) \cdot t \wedge \tau_m\right) = 1, \quad t \geq 0.$$

(iii) Now suppose $\mu \geq 0$. Show that for $\sigma > 0$,

$$\mathbb{E}\left[\exp\left(\sigma m - \left(\sigma\mu + \frac{\sigma^2}{2}\right)\tau_m\right) \cdot \mathbf{1}_{\{\tau_m < +\infty\}}\right] = 1.$$

(iv) Use this fact to show $\mathbb{P}(\tau_m < +\infty) = 1$ and to obtain the Laplace transform

$$\mathbb{E}e^{-\alpha\tau_m} = e^{m\mu - m\sqrt{2\alpha + \mu^2}} \quad \forall \alpha > 0.$$

Proof

(i) Notice that since

$$Z(t) := e^{\sigma W(t) - \frac{\sigma^2}{2}t}$$

the result follows from Theorem 3.6.1.

(ii) Since $Z(t)$ is a martingale and stopped martingale is again a martingale, we have

$$1 = Z(0) = \mathbb{E}[Z(t \wedge \tau_m)] = \mathbb{E}\left[\exp\left(\sigma \cdot X(t \wedge \tau_m) - \left(\sigma\mu + \frac{\sigma^2}{2}\right) \cdot t \wedge \tau_m\right)\right]$$

Discussion inside Theorem 3.6.1. shows that for any $\beta > 0$, the following is true

$$\lim_{t \to +\infty} \exp\left(-\beta \cdot t \wedge \tau_m\right) = \mathbf{1}_{\{\tau_m < +\infty\}} \cdot \exp\left(-\beta \cdot \tau_m\right).$$

Since we assume that $\mu \geq 0$, $\beta = \mu + \frac{\sigma^2}{2} > 0$, and thus

$$\lim_{t \to +\infty} \exp\left(-\left(\sigma\mu + \frac{\sigma^2}{2}\right) \cdot t \wedge \tau_m\right) = \mathbf{1}_{\{\tau_m < +\infty\}} \cdot \exp\left(-\left(\sigma\mu + \frac{\sigma^2}{2}\right) \cdot \tau_m\right).$$

The following is always true

$$\mathbb{E}\left[e^{\sigma \cdot X(t \wedge \tau_m)}\right] \leq e^{\sigma \cdot m}$$

Moreover, it holds that

$$\lim_{t\to+\infty} \exp\left(\sigma X(t\wedge\tau_m)\right) = e^{\sigma m} \qquad \text{on } \tau_m < +\infty$$

Putting pieces together, we have that

$$\lim_{t\to+\infty} \exp\left(\sigma X(t\wedge\tau_m) - \left(\sigma\mu + \frac{\sigma^2}{2}\right)\cdot t\wedge\tau_m\right) = \mathbf{1}_{\{\tau_m<+\infty\}}\cdot e^{\sigma m - \left(\sigma\mu+\frac{\sigma^2}{2}\right)\tau_m}$$

Appealing to Dominated Convergence Theorem, we have that

$$1 = \mathbb{E}\left[\mathbf{1}_{\{\tau_m<+\infty\}}\cdot e^{\sigma m-\left(\sigma\mu+\frac{\sigma^2}{2}\right)\tau_m}\right] \Rightarrow e^{-\sigma m} = \mathbb{E}\left[\mathbf{1}_{\{\tau_m<+\infty\}}\cdot e^{-\left(\sigma\mu+\frac{\sigma^2}{2}\right)\tau_m}\right] \tag{1}$$

(iii) Taking limit $\sigma\to 0$ inside (1), we obtain that

$$\mathbb{E}\left[\mathbf{1}_{\{\tau_m<+\infty\}}\right] = 1 \Rightarrow \mathbb{P}\left(\tau_m < +\infty\right) = 1.$$

Therefore, simplifying Eq (1), we get

$$e^{-\sigma m} = \mathbb{E}\left[e^{-\left(\sigma\mu+\frac{\sigma^2}{2}\right)\tau_m}\right]$$

We now obtain Laplace transform of the random variable τ_m. Recall that $m>0$. let $\alpha = \sigma\mu + \frac{\sigma^2}{2}$. Thus,

$$2\alpha + \mu^2 = (\sigma+\mu)^2 \Rightarrow \sigma+\mu = \sqrt{2\alpha+\mu^2}.$$

Notice that here we used the assumption that $\mu \geq 0$. Rewriting the last displayed equation, we conclude the result:

$$\mathbb{E}e^{-\alpha\tau_m} = e^{m\mu - m\sqrt{2\alpha+\mu^2}} \quad \forall\alpha > 0. \tag{2}$$

(iv) Taking derivative with respect to α from Eq (2), we have that

$$\mathbb{E}\left[\tau_m e^{-\alpha\tau_m}\right] = \frac{m e^{m\mu-m\sqrt{2\alpha+\mu^2}}}{\sqrt{2\alpha+\mu^2}}$$

Sending $\alpha\to 0$, we obtain that

$$\mathbb{E}[\tau_m] = \frac{m}{\mu}.$$

(v) Eq (1) still holds under the assumption that $\sigma > -2\mu$. Indeed, the proof only utilizes the fact that $\sigma\mu + \frac{\sigma^2}{2} \geq 0$. Notice that since τ_m can attain infinity with positive probability, we cannot drop the indicator factor $\mathbf{1}_{\{\tau_m<+\infty\}}$ in Eq (1). Moreover,

$$\mathbb{E}e^{-\alpha\tau_m} = \mathbb{E}e^{-\alpha\tau_m}\cdot\mathbf{1}_{\{\tau_m<+\infty\}} + \mathbb{E}e^{-\alpha\tau_m}\cdot\mathbf{1}_{\{\tau_m=+\infty\}} = \mathbb{E}e^{-\alpha\tau_m}\cdot\mathbf{1}_{\{\tau_m<+\infty\}}$$

Furthermore, in this case we cannot let $\sigma\to 0$ due to the fact that $\sigma > 2|\mu|$ which is strictly positive. However, we will let $\sigma \to \frac{-\mu}{2}$ instead to get:

$$e^{\frac{m\mu}{2}} = \mathbb{E}\left[\mathbf{1}_{\{\tau_m<+\infty\}}\right] = \mathbb{P}\left(\tau_m < +\infty\right).$$

The proof is complete.