Exercise 3.8

Denote

$$
\begin{aligned}
u_n &= e^{\frac{\sigma}{\sqrt{n}}} \\
d_n &= e^{-\frac{\sigma}{\sqrt{n}}} \\
\tilde{p}_n &= \frac{\frac{r}{n} + 1 - e^{-\frac{\sigma}{\sqrt{n}}}}{e^{\frac{\sigma}{\sqrt{n}}} - e^{-\frac{\sigma}{\sqrt{n}}}} \\
\tilde{q}_n &= \frac{e^{\frac{\sigma}{\sqrt{n}}} - \frac{r}{n} - 1}{e^{\frac{\sigma}{\sqrt{n}}} - e^{-\frac{\sigma}{\sqrt{n}}}} \\
M_{nt,n} &= \sum_{k=1}^{nt} X_{k,n} \\
S_n(t) &= S(0) u_n^{\frac{1}{2}(nt - M_{nt,n})} d_n^{\frac{1}{2}(nt + M_{nt,n})} \\
&= S(0) e^{\frac{\sigma}{\sqrt{n}} M_{nt,n}}
\end{aligned}
$$

Show that the limiting distribution of $S_n(t)$ (*i.e.*, stock price at time t with nt steps in binomial model) is the same as the distribution of a geometric Brownian motion $S(0)e^{\sigma W(t) + (r - \frac{\sigma^2}{2})t}$.

Proof

We begin by noting that

$$
\begin{aligned}
M_{\frac{1}{\sqrt{n}} M_{nt,n}}(u) &= \tilde{\mathbb{E}}\left[e^{\frac{u}{\sqrt{n}} M_{nt,n}} \right] \\
&= S(0) \Pi_{k=1}^{nt} \tilde{\mathbb{E}}\left[e^{\frac{u}{\sqrt{n}} X_{k,n}} \right] \\
&= S(0) \Pi_{k=1}^{nt} \left[\tilde{p}_n e^{\frac{u}{\sqrt{n}}} + \tilde{q}_n e^{-\frac{u}{\sqrt{n}}} \right] \\
&= S(0) \left[\tilde{p}_n e^{\frac{u}{\sqrt{n}}} + \tilde{q}_n e^{-\frac{u}{\sqrt{n}}} \right]^{nt} \\
&= \frac{t}{x^2} \log \underbrace{\left[\frac{(rx^2 + 1 - e^{-\sigma x}) e^{ux} - (rx^2 + 1 - e^{\sigma x}) e^{-ux}}{e^{\sigma x} - e^{-\sigma x}} \right]}_{:= f(x)}
\end{aligned}
$$

Here $x = \frac{1}{\sqrt{n}}$. We need the following fact below.

$$
\frac{e^{ax} - e^{-ax}}{e^{bx} - e^{-bx}} = \frac{a}{b} + \frac{x^2(a^3 - ab^2)}{6b} + \mathcal{O}(x^4).
$$

To see this, note that

$$\begin{aligned}
\frac{e^{ax}-e^{-ax}}{e^{bx}-e^{-bx}} &= \frac{2ax+\frac{1}{3}a^3x^3+\mathcal{O}(x^5)}{2bx+\frac{1}{3}b^3x^3+\mathcal{O}(x^5)} \\
&= \frac{2a+\frac{1}{3}a^3x^2+\mathcal{O}(x^4)}{2b+\frac{1}{3}b^3x^2+\mathcal{O}(x^4)} \\
&= \frac{2a+\frac{1}{3}a^3x^2}{2b+\frac{1}{3}b^3x^2}+\mathcal{O}(x^4) \\
&= \frac{a}{b}\cdot\frac{1+\frac{1}{6}a^2x^2}{1+\frac{1}{6}b^2x^2}+\mathcal{O}(x^4) \\
&= \frac{a}{b}\cdot\left(1+\frac{1}{6}a^2x^2\right)\left(1-\frac{1}{6}b^2x^2+\mathcal{O}(x^4)\right)+\mathcal{O}(x^4) \\
&= \frac{a}{b}\cdot\left(1+\frac{1}{6}a^2x^2-\frac{1}{6}b^2x^2\right)+\mathcal{O}(x^4)
\end{aligned}$$

Next,

$$\begin{aligned}
f(x) &= (1+rx^2)\cdot\frac{e^{ux}-e^{-ux}}{e^{\sigma x}-e^{-\sigma x}}-\frac{e^{(u-\sigma)x}-e^{(\sigma-u)x}}{e^{\sigma x}-e^{-\sigma x}} \\
&= (1+rx^2)\cdot\frac{u}{\sigma}\cdot\left(1+\frac{x^2}{6}\cdot\left(u^2-\sigma^2\right)\right)-\frac{u-\sigma}{\sigma}\cdot\left(1+\frac{x^2}{6}\cdot\left((u-\sigma)^2-\sigma^2\right)\right)+\mathcal{O}(x^4) \\
&= \frac{u}{\sigma}\cdot\left[1+\left(r+\frac{u^2-\sigma^2}{6}\right)x^2-\left(1-\frac{\sigma}{u}\right)-(u-\sigma)\cdot\frac{u-2\sigma}{6}x^2\right]+\mathcal{O}(x^4) \\
&= 1+\frac{u}{\sigma}\cdot\left(r+\frac{u^2-\sigma^2-(u-\sigma)(u-2\sigma)}{6}\right)x^2+\mathcal{O}(x^4) \\
&= 1+\frac{u}{\sigma}\cdot\left(r+\frac{\sigma u-\sigma^2}{2}\right)x^2+\mathcal{O}(x^4)
\end{aligned}$$

Using $\log(1+z)=z+\mathcal{O}(z^2)$, we obtain that

$$\log f(x)=\frac{u}{\sigma}\cdot\left(r+\frac{\sigma u-\sigma^2}{2}\right)x^2+\mathcal{O}(x^4)$$

Hence,

$$\lim_{x\downarrow 0}\frac{t}{x^2}f(x)=\frac{tu}{\sigma}\cdot\left(r+\frac{\sigma u-\sigma^2}{2}\right)$$

Change of variable $u\mapsto\sigma u$, we have that

$$\begin{aligned}
\lim_{n\to+\infty}M_{\frac{\sigma}{\sqrt{n}}M_{nt,n}}(u) &= e^{rtu+\frac{t\sigma^2u(u-1)}{2}} \\
&= e^{t(r-\frac{\sigma^2}{2})u+\frac{1}{2}t\sigma^2u}
\end{aligned}$$

The right hand side is the moment generating function of a normal distribution with mean $t(r-\frac{\sigma^2}{2})$ and variance $t\sigma^2$. Remember that $W(t)\sim\mathcal{N}(0,t)$.