Exercise 3.9 (Laplace transform of first passage density)

Consider a Brownian motion $W(t)$ without drift and for $m > 0$, denote its first passage density by

$$f(t,m) = \frac{m}{t\sqrt{2\pi t}} \exp\left(-\frac{m^2}{2t}\right)$$

Laplace transform of $f(t,m)$ is calculated as below

$$g(\alpha,m) = \int_0^{+\infty} e^{-\alpha t} f(t,m)\mathrm{d}t.$$

Prove that $g(\alpha,m) = e^{-m\sqrt{2\alpha}}$.

Proof

Throughout, assume that $k \geq 3$. Denote by

$$a_k(m) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} t^{-\frac{k}{2}} e^{-\alpha t - \frac{m^2}{2t}} \mathrm{d}t.$$

Therefore, $g(\alpha,m) = m a_3(m)$. Note

$$\begin{aligned}
\frac{\partial}{\partial m} a_k(m) &= \frac{\partial}{\partial m} \frac{1}{\sqrt{2\pi}} \int_0^{\infty} t^{-\frac{k}{2}} e^{-\alpha t - \frac{m^2}{2t}} \mathrm{d}t \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} t^{-\frac{k}{2}} \frac{\partial}{\partial m} e^{-\alpha t - \frac{m^2}{2t}} \mathrm{d}t \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} t^{-\frac{k}{2}} \left[-\frac{m}{t}\right] e^{-\alpha t - \frac{m^2}{2t}} \mathrm{d}t \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} t^{-\frac{k}{2}} \left[-\frac{m}{t}\right] e^{-\alpha t - \frac{m^2}{2t}} \mathrm{d}t \\
&= -\frac{m}{\sqrt{2\pi}} \int_0^{\infty} t^{-\frac{k+2}{2}} e^{-\alpha t - \frac{m^2}{2t}} \mathrm{d}t \\
&= -m a_{k+2}(m).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\frac{\partial}{\partial m} g(\alpha,m) &= a_3(m) + m \frac{\partial}{\partial m} a_3(m) \\
&= a_3(m) - m^2 a_5(m),
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2}{\partial m^2} g(\alpha,m) &= \frac{\partial}{\partial m} a_3(m) - \frac{\partial}{\partial m} m^2 a_5(m) \\
&= -m a_5(m) - 2m a_5(m) + m^3 a_7(m) \\
&= -3m a_5(m) + m^3 a_7(m).
\end{aligned}$$

Let $v = t^{-\frac{k-2}{2}}$. Then $\mathrm{d}v = -\frac{k-2}{2} \cdot t^{-\frac{k}{2}}$.

$$\begin{aligned}\frac{1}{\sqrt{2\pi}} \int_0^\infty \underbrace{e^{-\alpha t - \frac{m^2}{2t}}}_{u} \underbrace{t^{-\frac{k}{2}} \mathrm{d}t}_{= -\frac{2}{k-2} \mathrm{d}v} &= \frac{2}{\sqrt{2\pi}(2-k)} \int_0^\infty u \mathrm{d}v \\ &= \frac{2}{\sqrt{2\pi}(k-2)} \left[\int_0^\infty v \mathrm{d}u - uv|_0^\infty \right]\end{aligned}$$

Next,

$$\begin{aligned}0 \leq (uv)(0) &= \lim_{t \to 0^+} \frac{e^{-\alpha t - \frac{m^2}{2t}}}{\sqrt{t^{k-2}}} \\ &\leq \lim_{t \to 0^+} \frac{e^{-\frac{m^2}{2t}}}{\sqrt{t^{k-2}}}.\end{aligned}$$

We use the following inequality to prove that $(uv)(0) = 0$.

$$\frac{1}{x} + \log x \geq 1 \quad \forall x > 0.$$

We claim there exists constant $c > 0$ such that $\frac{e^{-\frac{m^2}{2t}}}{\sqrt{t^{k-2}}} \leq c\sqrt{t}$. Note

$$\begin{aligned}\frac{e^{-\frac{m^2}{2t}}}{\sqrt{t^{k-2}}} \leq c\sqrt{t} &\iff e^{-\frac{m^2}{2t}} \leq c\sqrt{t^{k-1}} \\ &\iff -\frac{m^2}{2t} \leq \log c + \frac{k-1}{2} \log t \\ &\iff 0 \leq \log c^{\frac{2}{k-1}} + \frac{m^2}{(k-1)t} + \log t \\ &\iff 0 \leq \log c - \log \frac{k-1}{m^2} + \underbrace{\frac{m^2}{(k-1)t}}_{= \frac{1}{x}} + \underbrace{\log \frac{(k-1)t}{m^2}}_{= x}\end{aligned}$$

So it suffices to let $c = \frac{k-1}{m^2}$. Therefore,

$$0 \leq (uv)(0) \leq \lim_{t \to 0^+} \frac{e^{-\frac{m^2}{2t}}}{\sqrt{t^{k-2}}} \leq \lim_{t \to 0^+} c\sqrt{t} = 0.$$

Moreover,

$$0 \leq (uv)(\infty) = \lim_{t \to \infty} \frac{e^{-\alpha t - \frac{m^2}{2t}}}{\sqrt{t^{k-2}}} \leq \lim_{t \to \infty} \frac{1}{\sqrt{t^{k-2}}} = 0.$$

Note that

$$\mathrm{d}u = \left(-\alpha + \frac{m^2}{2t^2} \right) u \mathrm{d}t$$

Thus,

$$\begin{aligned}\int_0^\infty v\mathrm{d}u &= \int_0^\infty t^{-\frac{k-2}{2}} \left(-\alpha + \frac{m^2}{2t^2}\right) e^{-\alpha t - \frac{m^2}{2t}} \mathrm{d}t \\ &= -\alpha \int_0^\infty t^{-\frac{k-2}{2}} e^{-\alpha t - \frac{m^2}{2t}} \mathrm{d}t + \frac{m^2}{2} \int_0^\infty t^{-\frac{k+2}{2}} e^{-\alpha t - \frac{m^2}{2t}} \mathrm{d}t \\ &= \sqrt{2\pi} \left(-\alpha a_{k-2}(m) + \frac{m^2}{2} a_{k+2}(m)\right).\end{aligned}$$

Putting pieces together,

$$a_k(m) = \frac{2}{k-2} \left(-\alpha a_{k-2}(m) + \frac{m^2}{2} a_{k+2}(m)\right).$$

Let $k = 5$ to obtain

$$a_5(m) = \frac{2}{3} \left(-\alpha a_3(m) + \frac{m^2}{2} a_7(m)\right)$$

In other words,

$$m^2 a_7(m) = 3a_5(m) + 2\alpha a_3(m)$$

Therefore,

$$\begin{aligned}\frac{\partial^2}{\partial m^2} g(\alpha, m) &= -3ma_5(m) + +m^3 a_7(m) \\ &= -3ma_5(m) + 3ma_5(m) + 2\alpha m a_3(m) \\ &= 2\alpha m a_3(m) \\ &= 2\alpha g(\alpha, m).\end{aligned}$$

Thus $g(\alpha, m)$ for each fixed α satisfies a second order differential equation. As a result,

$$g(\alpha, m) = A_1(\alpha) e^{m\sqrt{2\alpha}} + A_2(\alpha) e^{-m\sqrt{2\alpha}}.$$

We next show that

$$\lim_{m \to +\infty} g(\alpha, m) = 0.$$

This will immediately results that $A_1(\alpha) = 0$. We have that

$$\begin{aligned}g(\alpha, m) &= \int_0^{+\infty} e^{-\alpha t} \cdot \frac{m}{t\sqrt{2\pi t}} \exp\left(-\frac{m^2}{2t}\right) \mathrm{d}t \\ &= \int_0^{+\infty} e^{-\alpha m^2 t} \cdot \frac{m}{m^2 u \sqrt{2\pi m^2 u}} \exp\left(-\frac{m^2}{2m^2 u}\right) m^2 \mathrm{d}u \\ &= \int_0^{+\infty} e^{-\alpha m^2 t} \cdot \frac{1}{u\sqrt{2\pi u}} \exp\left(-\frac{1}{2u}\right) \mathrm{d}u.\end{aligned}$$

Using Dominated Convergence Theorem, it suffices to show that

$$\int_0^{+\infty} \frac{1}{u\sqrt{2\pi u}} \exp\left(-\frac{1}{2u}\right) \mathrm{d}u < +\infty.$$

However,

$$\begin{aligned}\int_0^{+\infty} \frac{1}{u\sqrt{2\pi u}} \exp\left(-\frac{1}{2u}\right) \mathrm{d}u &= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} x^3 \exp\left(-\frac{x^2}{2}\right) \frac{2}{x^3} \mathrm{d}x \\ &= \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \\ &= 1.\end{aligned}$$

Thus, $\lim_{m\to+\infty} g(\alpha, m) = 0$. We have also simultaneously showed that

$$\lim_{m\to 0^+} g(\alpha, m) = 1.$$

Therefore, $A_2(\alpha) = 1$. Proof is complete.