Exercise 4.1

Suppose that $M(t)$ is a martingale w.r.t $\mathcal{F}(t)$ for $0 \leq t \leq T$. Suppose that $\Delta(t)$ is a simple process adapted to $\mathcal{F}(t)$ for partition $\Pi = \{t_0, \cdots, t_n\}$ of $[0, T]$. In other words, $\Delta(t_j)$ is $\mathcal{F}(t_j)$-measurable and $\Delta(t)$ is constant in t on $[t_j, t_{j+1})$. Assuming that $M(t)$ represents price and an asset at time t, the capital gains accrue to the investor between times 0 and t is denoted by $I(t)$:

$$I(t) := \sum_{j=0}^{k-1} \Delta(t_j) \cdot [M(t_{j+1}) - M(t_j)] + \Delta(t_k) \cdot [M(t) - M(t_k)] \text{ where } t \in [t_k, t_{k+1}).$$

Show that $I(t)$ is a martingale for $0 \leq t \leq T$.

Proof

The proof follows from the proof of Theorem 4.2.1 *verbatim*. Note that in its proof Property (iii) of Definition 3.3.3 *i.e.,* Independence of future increments for Brownian motion is not used. Additionally, Property (i) and (ii) also holds here for $\mathcal{F}(t)$ and $M(t)$.