Exercise 4.10 (Self-financing trading)

Consider a self-financing strategy wherein the agent invests in the underlying asset *i.e.,* a stock and the money market account. Let $X(t)$ denote the value of the portfolio at time t and $\Delta(t)$ the number of shares of stock held at time t. The following then holds

$$\mathrm{d}X(t) = \Delta(t)\mathrm{d}S(t) + r\left(X(t) - \Delta(t)S(t)\right)\mathrm{d}t.$$

Denote by $\Gamma(t)$ and $M(t) = e^{rt}$ the number of shares and the price of a share of money market account held at time t respectively.

(i) Show that the continuous-time self-financing condition holds.

$$S(t)\mathrm{d}\Delta(t) + \mathrm{d}S(t)\mathrm{d}\Delta(t) + M(t)\mathrm{d}\Gamma(t) + \mathrm{d}M(t)\mathrm{d}\Gamma(t) = 0$$

(ii) Consider a portfolio that consists of a long call and short $\Delta(t)$ shares of stock. Value of this portfolio is denoted by $N(t)$ and it holds that $N(t) = c(t, S(t)) - \Delta(t)S(t)$. Assume that $X(0) = c(0, S(0))$ and let $\Delta(t) = c_x(t, S(t))$. We know already that $X(t) = c(t, S(t))$. Derive

$$rN(t) = c_t(t, S(t)) + \tfrac{1}{2}\sigma^2 S^2(t)c_{xx}(t, S(t))$$

Proof

1. Total value of portfolio at time t is equal to

$$X(t) = \Delta(t)S(t) + \Gamma(t)M(t).$$

Continuous-time self-financing condition holds iff

$$\mathrm{d}\Delta(t)S(t) + \mathrm{d}M(t)\Gamma(t) = \Delta(t)\mathrm{d}S(t) + \Gamma(t)\mathrm{d}M(t).$$

This holds iff

$$\mathrm{d}X(t) = \Delta(t)\mathrm{d}S(t) + \Gamma(t)\mathrm{d}M(t).$$

Using $\mathrm{d}X(t) = \Delta(t)\mathrm{d}S(t) + r\left(X(t) - \Delta(t)S(t)\right)\mathrm{d}t$, this holds iff

$$r\left(X(t) - \Delta(t)S(t)\right)\mathrm{d}t = \Gamma(t)\mathrm{d}M(t)$$

Since $X(t) - \Delta(t)S(t) = \Gamma(t)M(t)$, this holds iff

$$r\Gamma(t)M(t)\mathrm{d}t = \Gamma(t)\mathrm{d}M(t).$$

Finally, $M(t) = e^{rt}$, $\mathrm{d}M(t) = rM(t)\mathrm{d}t$.

2. Amount invested in money market account is equal to

$$X(t) - \Delta(t)S(t) = c(t, S(t)) - \Delta(t)S(t) = N(t) = \Gamma(t)M(t).$$

Since $N(t) = c(t, S(t)) - \Delta(t)S(t)$, it holds that

$$\begin{aligned}\mathrm{d}N(t) = {} & c_t(t, S(t))\mathrm{d}t + c_x(t, S(t))\mathrm{d}S(t) + \tfrac{1}{2}c_{xx}(t, S(t))\mathrm{d}S(t)\mathrm{d}S(t) \\ & - S(t)\mathrm{d}\Delta(t) - \Delta(t)\mathrm{d}S(t) - \mathrm{d}S(t)\mathrm{d}\Delta(t).\end{aligned}$$

Since $c_x(t, S(t))\mathrm{d}S(t) = \Delta(t)\mathrm{d}S(t)$, continuous-time self-financing condition gives

$$\begin{aligned}\mathrm{d}N(t) &= c_t(t, S(t))\mathrm{d}t + \tfrac{1}{2}c_{xx}(t, S(t))\mathrm{d}S(t)\mathrm{d}S(t) - S(t)\mathrm{d}\Delta(t) - \mathrm{d}S(t)\mathrm{d}\Delta(t) \\ &= c_t(t, S(t))\mathrm{d}t + \tfrac{1}{2}c_{xx}(t, S(t))\mathrm{d}S(t)\mathrm{d}S(t) + M(t)\mathrm{d}\Gamma(t) + \mathrm{d}M(t)\mathrm{d}\Gamma(t)\end{aligned}$$

Thus,

$$\begin{aligned}c_t(t, S(t))\mathrm{d}t + \tfrac{1}{2}c_{xx}(t, S(t))\mathrm{d}S(t)\mathrm{d}S(t) &= \mathrm{d}N(t) - M(t)\mathrm{d}\Gamma(t) - \mathrm{d}M(t)\mathrm{d}\Gamma(t) \\ &= \Gamma(t)\mathrm{d}M(t) \\ &= rM(t)\Gamma(t) \\ &= rN(t)\mathrm{d}t\end{aligned}$$

It immediately follows that

$$rN(t) = c_t(t, S(t)) + \tfrac{1}{2}\sigma^2 S^2(t)c_{xx}(t, S(t))$$