Exercise 4.11

Denote by $c(t,x)$ the market price for a European call with expiry T and strike price K under the assumption that the underlying stock follows a geometric Brownian motion with volatility σ_1. In this exercise, we consider the case where this assumption is *wrong* in the sense that the true volatility for the underlying asset is indeed $\sigma_2 \neq \sigma_1$. In other words, the following holds

$$\mathrm{d}S(t) = \alpha S(t)\mathrm{d}t + \sigma_2 S(t)\mathrm{d}W(t).$$

Suppose that $\sigma_2 > \sigma_1$. We construct a portfolio $X(t)$ with $X(t) = 0$ for all $t \in [0,T]$ which also exhibits an arbitrage opportunity. Portfolio $X(t)$ is described as below:

- $X(0) = 0$
- Long one European call
- Short $c_x(t, S(t))$ shares of stock
- Cash position *i.e.,* $X(t) - c(t, S(t)) + S(t)c_x(t, S(t))$ investment in money market at rate r
- Cash removal at rate $\frac{1}{2}\left(\sigma_2^2 - \sigma_1^2\right) S^2(t)c_{xx}(t, S(t))$

The following differential equation then holds for $t \in [0,T]$

$$\mathrm{d}X(t) = \mathrm{d}c(t,S(t)) - c_x(t,S(t))\mathrm{d}S(t) + r\left[X(t) - c(t,S(t)) + S(t)c_x(t,S(t))\right]\mathrm{d}t - \frac{1}{2}\left(\sigma_2^2 - \sigma_1^2\right) S^2(t)c_{xx}(t,S(t))\mathrm{d}t$$

Show that $X(t) = 0$ for $t \in [0,T]$ and explain the arbitrage opportunity thereof.

Proof

Gamma is always positive and therefore $c_{xx}(t, S(t)) > 0$. Consequently, once we show $X(t) = 0$ for $t \in [0,T]$ the arbitrage opportunity becomes apparent *i.e.,* we remove cash at a positive rate between time 0 and T with zero liability consistently. To see $X(t) = 0$ for $t \in [0,T]$, note that

$$\mathrm{d}e^{-rt}X(t) = -re^{-rt}X(t)\mathrm{d}t + e^{-rt}\mathrm{d}X(t) + re^{-rt}\underbrace{\mathrm{d}X(t)\mathrm{d}t}_{=0}$$

Furthermore, we have that

$$\begin{aligned}
\mathrm{d}c(t,S(t)) &= c_t(t,S(t))\mathrm{d}t + c_x(t,S(t))\mathrm{d}S(t) + \tfrac{1}{2}c_{xx}(t,S(t))\mathrm{d}S(t)\mathrm{d}S(t) \\
&= c_t(t,S(t))\mathrm{d}t + c_x(t,S(t))\mathrm{d}S(t) + \tfrac{1}{2}c_{xx}(t,S(t))\sigma_2^2 S(t)^2\mathrm{d}t
\end{aligned}$$

We conclude that

$$\begin{aligned}
\mathrm{d}X(t) &= c_t(t,S(t))\mathrm{d}t + r\left[X(t) - c(t,S(t)) + S(t)c_x(t,S(t))\right]\mathrm{d}t + \tfrac{1}{2}\sigma_1^2 S^2(t)c_{xx}(t,S(t))\mathrm{d}t \\
&= \underbrace{\left[c_t(t,S(t)) + rS(t)c_x(t,S(t)) + \tfrac{1}{2}\sigma_1^2 S^2(t)c_{xx}(t,S(t))\right]}_{=rc(t,S(t))\ \textit{i.e.,}\ \text{BSM partial differential eq.}}\mathrm{d}t + r\left[X(t) - c(t,S(t))\right]\mathrm{d}t \\
&= rX(t)\mathrm{d}t.
\end{aligned}$$

Putting pieces together, we obtain that

$$\mathrm{d}e^{-rt}X(t) = 0.$$

By assumption $X(0) = 0$. The result follows immediately.