Exercise 4.12

Denote by $\tau = T - t$ time to maturity. The Greeks are defined as follows.

- **Delta**

$$c_x(t,x) = N(d_+(\tau,x))$$

- **Theta**

$$c_t(t,x) = -rKe^{-r\tau}N(d_-(\tau,x)) - \frac{\sigma x}{2\sqrt{\tau}}N'(d_+(\tau,x))$$

- **Gamma**

$$c_{xx}(t,x) = N'(d_+(\tau,x))\frac{\partial}{\partial x}d_+(\tau,x) = \frac{1}{\sigma x\sqrt{\tau}}N'(d_+(\tau,x))$$

The value of a forward contract with strike price K is

$$f(t,x) = x - e^{-r\tau}K.$$

Also, the value of a forward contract at time t agrees with the value of the portfolio that is long one call and short one put. In other words,

$$f(t,x) = c(t,x) - p(t,x) \quad \forall x \geq 0.$$

(i) Derive Greeks for a European put, *i.e.*, $p_x(t,x), p_t(t,x)$ and $p_{xx}(t,x)$.

(ii) An agent is hedging a short position in a put. Show that she should have a short position in the underlying stock and a long position in the money market. Moreover, show that both $f(t,x)$ and $p(t,x)$ satisfies the same BSM PDE that is satisfied by $c(t,x)$.

Proof

(i) Notice that $N(u) + N(-u) = 1$. We have that

- **Delta**

$$\begin{aligned} p_x(t,x) &= c_x(t,x) - f_x(t,x) \\ &= N(d_+(\tau,x)) - 1 \\ &= -N(-d_+(\tau,x)) \end{aligned}$$

- **Theta**

$$\begin{aligned} p_t(t,x) &= c_t(t,x) - f_t(t,x) \\ &= -rKe^{-r\tau}N(d_-(\tau,x)) - \frac{\sigma x}{2\sqrt{\tau}}N'(d_+(\tau,x)) + re^{-\tau}K \\ &= rKe^{-r\tau}N(-d_-(\tau,x)) - \frac{\sigma x}{2\sqrt{\tau}}N'(d_+(\tau,x)) \end{aligned}$$

- **Gamma**

$$\begin{aligned} p_{xx}(t,x) &= c_{xx}(t,x) - f_{xx}(t,x) \\ &= N'(d_+(\tau,x))\frac{\partial}{\partial x}d_+(\tau,x) = \frac{1}{\sigma x\sqrt{\tau}}N'(d_+(\tau,x)) \\ &= c_{xx}(t,x). \end{aligned}$$

(ii) Denote the agent's portfolio's value at time t by $X(t)$. Suppose that at each time the agent holds $\Delta(t)$ of shares (We want to show that $\Delta(t) < 0$). The reminder of the portfolio value is invested in the money market account. Differential of this portfolio's value is as follows

$$\mathrm{d}X(t) = \Delta(t)\mathrm{d}S(t) + r\left(X(t) - \Delta(t)S(t)\right)\mathrm{d}t.$$

To hedge a short put option, for each $t \in [0,T]$ must satisfy $X(t) = p(t,S(t))$. In particular, initial capital is equal to $p(0,S(0))$. It equivalently holds that

$$\mathrm{d}e^{-rt}X(t) = \mathrm{d}e^{-rt}p(t,S(t)) \quad \forall t \in [0,T) \tag{1}$$

Notice that the same calculation carried in Section 4.5.3 holds when $c(t,S(t))$ is replaced by $p(t,S(t))$. Therefore, $p(t,x)$ satisfies the BSM PDE as follows

$$p_t(t,x) + rxp_x(t,x) + \tfrac{1}{2}\sigma^2x^2p_{xx}(t,x) = rp(t,x) \quad \forall t \in [0,T), x \geq 0.$$

The terminal condition is

$$p(T,x) = (K - X)^+.$$

Moreover, the following must be true to ensure (1) holds.

$$\Delta(t) = p_x(t,S(t)) \quad \forall t \in [0,T).$$

But, we know that $p_x(t,S(t)) = -N(-d_+(\tau,S(t))) < 0$. Hence, agent has a short position in the stock. Continuing, rearranging BSM PDE gives

$$\begin{aligned} p_t(t,S(t)) + \tfrac{1}{2}\sigma^2S^2(t)p_{xx}(t,S(t)) &= r\left[p(t,S(t)) - S(t)p_x(t,S(t))\right] \\ &= r\left[X(t) - S(t)\Delta(t)\right] \\ &= r\Gamma(t)M(t). \end{aligned}$$

Here $\Gamma(t)$ and $M(t) = e^{rt}$ denotes the number of shares and the price of a share of money market account held at time t respectively. However,

$$\begin{aligned} p_t(t,S(t)) + \tfrac{1}{2}\sigma^2S^2(t)p_{xx}(t,S(t)) &= rKe^{-r\tau}N(-d_-(\tau,x)) - \frac{\sigma x}{2\sqrt{\tau}}N'(d_+(\tau,x)) + \frac{\sigma x}{2\sqrt{\tau}}N'(d_+(\tau,x)) \\ &= rKe^{-r\tau}N(-d_-(\tau,x)) > 0. \end{aligned}$$

Thus, $\Gamma(t) > 0$ and the agent is long on the money market account. It remains to show that $f(t,x)$ satifies the BSM PDE. To see this, note that

$$\begin{aligned} f_t(t,x) + rxf_x(t,x) + \tfrac{1}{2}\sigma^2x^2f_{xx}(t,x) &= \left[c_t(t,x) - p_t(t,x)\right] + rx\left[c_x(t,x) - p_x(t,x)\right] \\ &\quad + \tfrac{1}{2}\sigma^2x^2\left[c_{xx}(t,x) - p_{xx}(t,x)\right] \\ &= r\left[c(t,x) - p(t,x)\right] \\ &= rf(t,x). \end{aligned}$$

The terminal condition is equal to

$$f(T,x) = x - K.$$