Exercise 4.13 (Decomposition of correlated BMs into independent BMs)

Suppose $B_1(t)$ and $B_2(t)$ are Brownian motions and

$$\mathrm{d}B_1(t)\mathrm{d}B_2(t) = \rho(t)\mathrm{d}t.$$

Here ρ is a stochastic process taking values strictly between -1 and 1. Define processes $W_1(t)$ and $W_2(t)$ in such a way that

$$B_1(t) = W_1(t), B_2(t) = \int_0^t \rho(s)\mathrm{d}W_1(s) + \int_0^t \sqrt{1-\rho^2(s)}\mathrm{d}W_2(s)$$

and show that $W_1(1)$ and $W_2(t)$ are independent Brownian motion.

Proof

Note that the question is asking us to define $W_1(t)$ and $W_2(t)$ first. We do not need to show that if W_1 and W_2 satisfy the displayed equations, then they are Brownian motions.

$W_1(t)$ is, self-evidently, defined as $B_1(t)$. Now define

$$W_2(t) = \int_0^t \frac{1}{\sqrt{1-\rho^2(s)}} \cdot \mathrm{d}B_2(s) - \int_0^t \frac{\rho(s)}{\sqrt{1-\rho^2(s)}} \cdot \mathrm{d}B_1(s)$$

In fact, this equation is easily derived from statement's displayed equation. We need to assume that the following regularity condition holds

$$\mathbb{E}\int_0^t \frac{1}{1-\rho^2(s)}\mathrm{d}s < +\infty$$

Note that

$$\mathbb{E}\int_0^t \frac{\rho^2(s)}{1-\rho^2(s)}\mathrm{d}s = -t + \mathbb{E}\int_0^t \frac{1}{1-\rho^2(s)}\mathrm{d}s < +\infty$$

$W_2(t), t \geq 0$ is continuous and as sum of two Itô integral, it must be a martingale too. Continuing, we have that

$$\mathrm{d}W_2(t) \cdot \mathrm{d}W_2(t) = \frac{1}{1-\rho^2(t)} \cdot \left(1 + \rho^2(t) - 2\rho^2(t)\right)\mathrm{d}t = 1.$$

Furthermore,

$$\mathrm{d}W_1(t)\mathrm{d}W_2(t) = \frac{1}{\sqrt{1-\rho^2(t)}} \cdot (\rho(t) - \rho(t))\,\mathrm{d}t = 0.$$

Using Levy's theorem in dimension two, it follows that $W_1(t), t \geq 0$ and $W_2(t), t \geq 0$ are independent Brownian motions. We have that

$$\mathrm{d}W_2(t) = \frac{1}{\sqrt{1-\rho^2(t)}} \cdot \mathrm{d}B_2(t) - \frac{\rho(t)}{\sqrt{1-\rho^2(t)}} \cdot \mathrm{d}B_1(t)$$

Thus,

$$\mathrm{d}B_2(t) = \rho(t)\mathrm{d}W_1(t) + \sqrt{1-\rho^2(t)}\mathrm{d}W_2(t)$$

Since $B_2(0) = 0$, integration from both sides gives the result.